Partial derivatives and confidence intervals of bivariate tail dependence functions

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Abstract

Bivariate extreme value theory was used to estimate a rare event (see de Haan and de Ronde [1998. Sea and wind: multivariate extremes at work. Extremes 1, 7–45]). This procedure involves estimating a tail dependence function. There are several estimators for the tail dependence function in the literature, but their limiting distributions depend on partial derivatives of the tail dependence function. In this paper smooth estimators are proposed for estimating partial derivatives of bivariate tail dependence functions and their asymptotic distributions are derived as well. A simulation study is conducted to compare different estimators of partial derivatives in terms of both mean squared errors and coverage accuracy of confidence intervals of the bivariate tail dependence function based on these different estimators of partial derivatives.

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1. Introduction

Let \((X_1, Y_1), (X_2, Y_2), \ldots\) be independent random vectors with common distribution function \(F\). Suppose there exist \(a_n > 0, c_n > 0, b_n \in R\) and \(d_n \in R\) such that

\[
\lim_{n \to \infty} P \left( \frac{\sqrt{n} X_i - b_n}{a_n} \leq x, \frac{\sqrt{n} Y_i - d_n}{c_n} \leq y \right) = G(x, y)
\]

for \((x, y) \in R^2\), where \(G\) has non-degenerate marginals. In this case, we say that \(F\) is in the domain of attraction of an extreme value distribution \(G\). Without loss of generality we can assume that

\[
\begin{align*}
G_1(x) &= G(x, \infty) = \exp\{- (1 + \gamma_1 x)^{-1/\gamma_1}\}, \\
G_2(y) &= G(\infty, y) = \exp\{- (1 + \gamma_2 y)^{-1/\gamma_2}\}
\end{align*}
\]
for some \( \gamma_1, \gamma_2 \in R \) and \( 1 + \gamma_1 x > 0, 1 + \gamma_2 y > 0 \). It is easy to check that condition (1) can be divided into marginal conditions:
\[
\begin{align*}
\lim_{n \to \infty} n \{ 1 - F( a_n x + b_n ) \} &= - \log G_1(x), \\
\lim_{n \to \infty} n \{ 1 - F( c_n y + d_n ) \} &= - \log G_2(y)
\end{align*}
\]
(2)
and dependence condition
\[
\lim_{n \to \infty} n \{ 1 - F( Q_1(1 - x/n), Q_2(1 - y/n) ) \} = l(x, y)
\]
(3)
for \( x, y > 0 \), where \( Q_1(x) = F_1^{-}(x) \), \( Q_2(y) = F_2^{-}(y) \), \( F_1(x) = F(x, \infty) \), \( F_2(y) = F(\infty, y) \), \( (\cdot)^- \) denotes the generalized inverse function of \( (\cdot) \), and
\[
l(x, y) = - \log G(( - \log G_1)^-(x), ( - \log G_2)^-(y)).
\]
The function \( l(x, y) \) is called a stable tail dependence function by Huang (1992), which plays an important role in estimating a rare event (see de Haan and de Ronde, 1998; de Haan and Sinha, 1999).

Define
\[
F_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} I( X_i \leq x, Y_i \leq y),
\]
\[
F_{n1}(x) = \frac{1}{n} \sum_{i=1}^{n} I( X_i \leq x), \quad F_{n2}(y) = \frac{1}{n} \sum_{i=1}^{n} I( Y_i \leq y),
\]
and set
\[
Q_{n1}(x) = F_{n1}^{-}(x), \quad Q_{n2}(y) = F_{n2}^{-}(y).
\]
Based on (3), a natural estimator of \( l(x, y) \), defined as
\[
\hat{l}_n(x, y) = \frac{1}{m} \left[ 1 - F_n \left( Q_{n1} \left( 1 - \frac{m}{n} x \right), Q_{n2} \left( 1 - \frac{m}{n} y \right) \right) \right],
\]
(4)
is studied by Huang (1992). A nice property of this estimator is that it is free of marginals. Other alternative estimators include Einmahl et al. (1993) and de Haan and Resnick (1993). The optimal rate of convergence for estimating \( l(x, y) \) is given by Drees and Huang (1998). For estimating spectral measure instead of tail dependence function, we refer to Einmahl et al. (1997, 2001). For parametrically modeling dependence of extremes, i.e., modeling \( 1 - F( Q_1(1 - x), Q_2(1 - y) ) \), we refer to Tawn (1988) and Ledford and Tawn (1997).

To obtain a confidence interval for the tail dependence function \( l(x, y) \), an obvious approach is via the limiting distribution of an estimator of \( l(x, y) \). As far as we know, the limiting distributions of all existent estimators for \( l(x, y) \) depend on the first partial derivatives of \( l(x, y) \). Hence, a confidence interval based on the approximation method requires estimating the first partial derivatives. On the other hand, estimating partial derivatives has its own role in exploring data. In this paper, we first propose smooth estimators for the partial derivatives and then compare with the simple estimator in Peng (1999) and another one suggested by an associate editor via a simulation study.

The rest of this paper is organized as follows. In Section 2, our methodology and main result are given. A simulation study is given in Section 3. All proofs are put in Section 4.

2. Methodology and main result

It follows from (3) and the homogeneous property of the tail dependence function that
\[
1 - F( Q_1(1 - x/n), Q_2(1 - y/n) ) \approx l(x, y)/n = l(x/n, y/n),
\]
i.e.,
\[
1 - F(x, y) \approx l(1 - F_1(x), 1 - F_2(y)) \quad \text{as } x, y \to \infty,
\]
Then our estimators become, i.e.,
\[
\frac{n}{m} (1 - F(x, y)) \approx \frac{n}{m} l(1 - F_1(x), 1 - F_2(y)) = l \left( \frac{n}{m} (1 - F_1(x)), \frac{n}{m} (1 - F_2(y)) \right)
\]
as \(x, y \to \infty\). Further,
\[
\frac{n}{m} (1 - F_n(X_i, Y_i)) - \hat{l}_n(x, y) \approx l \left( \frac{n}{m} (1 - F_{n1}(X_i)), \frac{n}{m} (1 - F_{n2}(Y_i)) \right) - l(x, y)
\]
\[
\approx \frac{\partial l}{\partial x}(x, y) \left[ \frac{n}{m} (1 - F_{n1}(X_i)) - x \right] + \frac{\partial l}{\partial y}(x, y) \left[ \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right],
\]
where \(\hat{l}_n(x, y)\) is defined as in Eq. (4) in Section 1. Set
\[
l_1(x, y) = \frac{\partial}{\partial x} l(x, y) \quad \text{and} \quad l_2(x, y) = \frac{\partial}{\partial y} l(x, y).
\]
Hence we propose to estimate \(l_1(x, y)\) and \(l_2(x, y)\) via minimizing the following weighted least squares
\[
\sum_{i=1}^{n} \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \hat{l}_n(x, y) - a \left[ \frac{n}{m} (1 - F_{n1}(X_i)) - x \right] 
\right. 
\left. - b \left[ \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right] \right\}^2 k_h \left( \frac{n}{m} (1 - F_{n1}(X_i)) - x \right) k_h \left( \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right),
\]
where \(k_h(x) = k(x/h)\), \(k\) is a kernel, \(h = h(n) \to 0\) is a bandwidth, \(m = m(n) \to \infty\) and \(m/n \to 0\) as \(n \to \infty\). For simplicity let
\[
w_i = w(X_i, Y_i, h, m) = k_h \left( \frac{n}{m} (1 - F_{n1}(X_i)) - x \right) k_h \left( \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right).
\]
Put
\[
\begin{align*}
\{ d_1 & = \sum_{i=1}^{n} \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \hat{l}_n(x, y) \right\} \left[ \frac{n}{m} (1 - F_{n1}(X_i)) - x \right] w_i, \\
\{ d_2 & = \sum_{i=1}^{n} \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \hat{l}_n(x, y) \right\} \left[ \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right] w_i, \\
\{ D_1 & = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_{n1}(X_i)) - x \right]^2 w_i, \\
\{ D_2 & = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_{n1}(X_i)) - x \right] \left[ \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right] w_i, \\
\{ D_3 & = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_{n2}(Y_i)) - y \right]^2 w_i.
\end{align*}
\]
Then our estimators become
\[
\hat{l}_1(x, y) = \frac{d_1 D_3 - d_2 D_2}{D_1 D_3 - D_2^2} \quad \text{and} \quad \hat{l}_2(x, y) = \frac{d_2 D_1 - d_1 D_2}{D_1 D_3 - D_2^2}.
\]
In order to derive the asymptotic properties of \(\hat{l}_1(x, y)\) and \(\hat{l}_2(x, y)\), we need a stricter condition than (3): suppose there exists \(A(t) \to 0\) such that
\[
\lim_{t \to 0} \frac{\{1 - F(Q_1(1 - tx), Q_2(1 - ty))\}/t - l(x, y)}{A(t)} = c(x, y)
\]
holds uniformly on $\mathcal{F}={(x, y) : x^2 + y^2 = 1, x \geq 0, y \geq 0}$, where $c(x, y)$ is non-constant and not a multiple of $l(x, y)$.

Before giving the asymptotic limits, we list our regularity conditions as follows:

(A1) $m = m(n) \to \infty$, $m/n \to 0$ and $\sqrt{m}A(m/n) \to 0$ as $n \to \infty$;

(A2) $l(x, y)$ has continuous third partial derivatives and $(\partial^2/\partial x \partial y)l(x, y) \neq 0$;

(A3) $k(x)$ is a symmetric probability density function with support $(-1, 1)$, and $k'(x)$ is bounded;

(A4) $h \to 0$, $mh^3 \to \infty$ and $\sqrt{m}A(m/n)/\sqrt{h} \to 0$ as $n \to \infty$.

Our main result is the following.

**Theorem 1.** Suppose (5) and regularity conditions (A1)–(A4) hold. Then, for all $x > 0$, $y > 0$,

\[
\begin{aligned}
\sqrt{mh} \begin{cases}
\hat{l}_1(x, y) - l_1(x, y) \\
+ h^2 \frac{(1/6)(\partial^3/\partial x^3)l(x, y) \int s^4 k(s) \, ds + (1/2)(\partial^3/\partial x \partial y^2)l(x, y)(\int s^2 k(s) \, ds)^2}{\int s^2 k(s) \, ds}
\end{cases} \\
\sqrt{mh} \begin{cases}
\hat{l}_2(x, y) - l_2(x, y) \\
+ h^2 \frac{(1/6)(\partial^3/\partial y^3)l(x, y) \int s^4 k(s) \, ds + (1/2)(\partial^3/\partial x^2 \partial y)l(x, y)(\int s^2 k(s) \, ds)^2}{\int s^2 k(s) \, ds}
\end{cases}
\end{aligned}
\]

\[
d \to Z_1 \sim N(0, \sigma_1^2),
\]

\[
d \to Z_2 \sim N(0, \sigma_2^2),
\]

where $Z_1$ and $Z_2$ are independent,

$$\sigma_1^2 = \frac{2l_1(x, y) \int_0^1 \int_0^1 (s \wedge t)k(s)k(t) \, ds \, dt + 2l_2(x, y) \int_0^1 \int_0^1 (s \wedge t)stk(s)k(t) \, ds \, dt}{(\int s^2 k(s) \, ds)^2},$$

and

$$\sigma_2^2 = \frac{2l_1(x, y) \int_0^1 \int_0^1 (s \wedge t)stk(s)k(t) \, ds \, dt + 2l_2(x, y) \int_0^1 \int_0^1 (s \wedge t)k(s)k(t) \, ds \, dt}{(\int s^2 k(s) \, ds)^2}.$$

**Remark 1.** It follows from the proof of Lemma 1 below that the condition $\sqrt{m}A(m/n)/\sqrt{h} \to 0$ can be removed if we replace (5) by assuming that

$$\lim_{t \to 0} \frac{\partial^2}{\partial x \partial y} \left[ \frac{1 - F(Q_1(1-tx), Q_2(1-ty))}{A(t)} - l(x, y) \right] = c_1(x, y)$$

holds uniformly on $\mathcal{F}$.

**Remark 2.** It follows from Theorem 1 that we can choose $h = O(m^{-1/5})$ and then choose $m$ such that $m^{3/5}A(m/n) \to 0$.

### 3. Simulation study

In this section, we compare the smooth estimators in Section 2 with the following two types of estimators. The first type is the simple estimators proposed by Peng (1999), which are defined as

$$\hat{I}_1^p(x, y) = k^{1/4}[\hat{l}_n(x + k^{-1/4}, y) - \hat{l}_n(x, y)]$$

and

$$\hat{I}_2^p(x, y) = k^{1/4}[\hat{l}_n(x, y + k^{-1/4}) - \hat{l}_n(x, y)].$$
Another type of estimators is suggested by an associate editor and is defined as
\[ \hat{I}^{AE}_{1}(x, y) = \int_{\arctan(y/x)}^{\pi/2} \min(1, \tan(\theta)) \hat{\Phi}(d\theta) \]
and
\[ \hat{I}^{AE}_{2}(x, y) = \int_{0}^{\arctan(y/x)} \min(1, \cot(\theta)) \hat{\Phi}(d\theta), \]
where \( \hat{\Phi}(\theta) \) is the spectral measure estimator defined in Einmahl et al. (2001). Since the spectral measure estimators studied by Einmahl et al. (1997) and Einmahl et al. (2001) employ a stricter second order condition than (A1), it seems that the number of upper order statistics involved in estimating the spectral measure has a smaller order than \( \hat{I}_{n}(x, y) \). Hence it is hard to compare the rates of convergence of \( \hat{I}_{1}(x, y) \) and \( \hat{I}_{2}(x, y) \) with those of \( \hat{I}^{AE}_{1}(x, y) \) and \( \hat{I}^{AE}_{2}(x, y) \) theoretically. Instead, we conduct the following simulation study.

We draw 1000 random samples of size \( n = 1000 \) from an elliptical vector \( RAU \), where
\[
A = \left( \begin{array}{cc}
\sqrt{1 + \rho} + \sqrt{1 - \rho} & \sqrt{1 - \rho} \\
\sqrt{1 + \rho} - \sqrt{1 - \rho} & \sqrt{1 + \rho} + \sqrt{1 - \rho}
\end{array} \right),
\]
\(-1 < \rho < 1, R > 0 \) is a random variable with distribution function \( \exp\{-x^{-1}\} \), \( U = (U_{1}, U_{2})^{T} \) is a random vector uniformly distributed on the unit sphere \( \{(u_{1}, u_{2}) : u_{1}^{2} + u_{2}^{2} = 1\} \), and \( U \) is independent of \( R \). It follows from Klüppelberg et al. (2005) that
\[
I(x, y) = x + y - \frac{x \int_{\pi/2}^{\pi/2} \cos(\phi) \, d\phi + y \int_{-\pi/2}^{\pi/2} g(x/y) \sin(\phi + \arcsin \rho) \, d\phi}{\int_{-\pi/2}^{\pi/2} \cos(\phi) \, d\phi},
\]
\[
l_{1}(x, y) = 1 - \frac{\int_{-\pi/2}^{\pi/2} g(x/y) \cos(\phi) \, d\phi}{\int_{-\pi/2}^{\pi/2} \cos(\phi) \, d\phi}, \quad \text{and} \quad l_{2}(x, y) = 1 - \frac{\int_{-\pi/2}^{\pi/2} g(x/y) \sin(\phi + \arcsin \rho) \, d\phi}{\int_{-\pi/2}^{\pi/2} \cos(\phi) \, d\phi},
\]
where \( g(t) = \arctan \left( (t - \rho)/\sqrt{1 - \rho^{2}} \right) \). We employ kernel \( k(x) = \frac{3}{4} (1 - x^{2}) I(|x| \leq 1) \) and \( h = m^{-1/5} \) in computing \( \hat{I}_{1}(x, y) \) and \( \hat{I}_{2}(x, y) \). In Table 1, we report the estimators \( \hat{I}_{1} \left( 0, \sqrt{1 - \theta^{2}} \right), \hat{I}_{2} \left( 0, \sqrt{1 - \theta^{2}} \right), \hat{I}_{1}^{p} \left( 0, \sqrt{1 - \theta^{2}} \right), \hat{I}_{2}^{p} \left( 0, \sqrt{1 - \theta^{2}} \right) \) for \( \theta = 0.2, 0.5, 0.8 \) and their corresponding square root of mean squared error, which show that these estimators are comparable and estimators with \( m = 100 \) have smaller mean squared errors than those with \( m = 50 \). However, \( \hat{I}_{1}(x, y) \) and \( \hat{I}_{2}(x, y) \) may be improved by choosing other bandwidths.

Next we consider the issue of constructing confidence intervals for \( I(x, y) \). Since
\[
\sqrt{m} \{ \hat{I}_{n}(x, y) - I(x, y) \} \overset{d}{\rightarrow} N(0, \sigma^{2}(x, y; l, l_{1}, l_{2})),
\]
where
\[
\sigma^{2}(x, y; l, l_{1}, l_{2}) = l(x, y) + x \tilde{l}_{1}^{2}(x, y) + y \tilde{l}_{2}^{2}(x, y) - 2xl_{1}(x, y) - 2yl_{2}(x, y) + 2l_{1}(x, y)l_{2}(x, y)(x + y - l(x, y))
\]
(see Huang, 1992), confidence intervals for \( I(x, y) \) based on different estimators of \( l_{1} \) and \( l_{2} \) are
\[
I_{x}(x, y) = (\hat{I}_{n}(x, y) - z_{m}m^{-1/2} \sigma^{2}(x, y, \hat{I}_{n}, \hat{I}_{1}, \hat{I}_{2}), \quad \hat{I}_{n}(x, y) + z_{m}m^{-1/2} \sigma^{2}(x, y, \hat{I}_{n}, \hat{I}_{1}, \hat{I}_{2}),
\]
\[
I_{x}^{p}(x, y) = (\hat{I}_{n}(x, y) - z_{m}m^{-1/2} \sigma^{2}(x, y, \hat{I}_{n}, \hat{I}_{1}^{p}, \hat{I}_{2}^{p}), \quad \hat{I}_{n}(x, y) + z_{m}m^{-1/2} \sigma^{2}(x, y, \hat{I}_{n}, \hat{I}_{1}^{p}, \hat{I}_{2}^{p})).
\]
Averages of estimators of partial derivatives of the tail dependence function \( \hat{l}(\theta, \sqrt{1 - \theta^2}) \) are given with corresponding square root of mean squared error in brackets.

<table>
<thead>
<tr>
<th>((\rho, m, \theta))</th>
<th>(\hat{l}_1)</th>
<th>(\hat{l}_1^P)</th>
<th>(\hat{l}_1^{AE})</th>
<th>(\hat{l}_2)</th>
<th>(\hat{l}_2^P)</th>
<th>(\hat{l}_2^{AE})</th>
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<td>0.570</td>
<td>0.478</td>
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<td>0.595</td>
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<td>0.655</td>
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</table>

The numbers denote the coverage probabilities for \( I_{0.95}(\theta, \sqrt{1 - \theta^2}) \), \( I_{0.95}^P(\theta, \sqrt{1 - \theta^2}) \) and \( I_{0.95}^{AE}(\theta, \sqrt{1 - \theta^2}) \)

<table>
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<th>((\rho, m, \theta))</th>
<th>(I_{0.95}(\theta, \sqrt{1 - \theta^2}))</th>
<th>(I_{0.95}^P(\theta, \sqrt{1 - \theta^2}))</th>
<th>(I_{0.95}^{AE}(\theta, \sqrt{1 - \theta^2}))</th>
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<td>(0.5, 100, 0.5)</td>
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<td>0.974</td>
<td>0.935</td>
</tr>
<tr>
<td>(0.5, 100, 0.8)</td>
<td>0.954</td>
<td>0.980</td>
<td>0.954</td>
</tr>
</tbody>
</table>

and

\[
I_{AE}^x(x, y) = (\hat{l}_n(x, y) - z_2 m^{-1/2} \tilde{\sigma}^2(x, y, \hat{l}_n, \hat{l}_1^{AE}, \hat{l}_2^{AE}), \quad \hat{l}_n(x, y) + z_2 m^{-1/2} \tilde{\sigma}^2(x, y, \hat{l}_n, \hat{l}_1^{AE}, \hat{l}_2^{AE})),
\]

where \(z_2\) satisfies \( P(|N(0, 1)| \leq z_2) = \alpha \). In Table 2, we report the coverage probabilities of these confidence intervals based on 1000 random samples. From Table 2, we observe that both \(I_x(x, y)\) and \(I_{AE}^x(x, y)\) are more accurate than \(I_x^P(x, y)\), but \(I_x(x, y)\) and \(I_{AE}^x(x, y)\) are comparable.
4. Proofs

Define

\[ A_{n, x, y}(s, t) = \sqrt{mh} \left\{ \frac{1}{mh} \sum_{i=1}^{n} \left[ I \left( X_i \leq Q_1 \left( 1 - \frac{m}{n} (x + sh) \right), Y_i \leq Q_2 \left( 1 - \frac{m}{n} (y + th) \right) \right) ight. \right. \]

\[ - I \left( X_i \leq Q_1 \left( 1 - \frac{m}{n} x \right), Y_i \leq Q_2 \left( 1 - \frac{m}{n} y \right) \right) \left. \right\} + \frac{l(x + sh, y + th) - l(x, y)}{h} \],

\[ \delta_1 = \sum_{i=1}^{n} \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \frac{n}{m} \left[ 1 - F_n \left( Q_{n1} \left( 1 - \frac{m}{n} x \right), Q_{n2} \left( 1 - \frac{m}{n} y \right) \right) \right] ight. \]

\[ - l \left( \frac{n}{m} [1 - F_n(X_i)], \frac{n}{m} [1 - F_n(Y_i)] \right) + l(x, y) \left[ \frac{n}{m} (1 - F_n(X_i)) - x \right] w_i, \]

\[ \delta_2 = \sum_{i=1}^{n} \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \frac{n}{m} \left[ 1 - F_n \left( Q_{n1} \left( 1 - \frac{m}{n} x \right), Q_{n2} \left( 1 - \frac{m}{n} y \right) \right) \right] ight. \]

\[ - l \left( \frac{n}{m} [1 - F_n(X_i)], \frac{n}{m} [1 - F_n(Y_i)] \right) + l(x, y) \left[ \frac{n}{m} (1 - F_n(Y_i)) - y \right] w_i, \]

\[ \delta_3 = \sum_{i=1}^{n} \left\{ l \left( \frac{n}{m} [1 - F_n(X_i)], \frac{n}{m} [1 - F_n(Y_i)] \right) - l(x, y) - l_1(x, y) \left[ \frac{n}{m} (1 - F_n(X_i)) - x \right] \right. \]

\[ - l_2(x, y) \left[ \frac{n}{m} (1 - F_n(Y_i)) - y \right] \left[ \frac{n}{m} (1 - F_n(X_i)) - x \right] w_i \]

and

\[ \delta_4 = \sum_{i=1}^{n} \left\{ l \left( \frac{n}{m} [1 - F_n(X_i) - x], \frac{n}{m} [1 - F_n(Y_i)] \right) - l(x, y) - l_1(x, y) \left[ \frac{n}{m} (1 - F_n(X_i)) - x \right] \right. \]

\[ - l_2(x, y) \left[ \frac{n}{m} (1 - F_n(Y_i)) - y \right] \left[ \frac{n}{m} (1 - F_n(Y_i)) - y \right] w_i. \]

Before we prove Theorem 1, we show the following lemmas.

Lemma 1. Under the conditions of Theorem 1, for any fixed \( x > 0 \) and \( y > 0 \), there exists a continuous Wiener process \( W_{x, y}(s, t) \) such that

\[ \sup_{-1 \leq s, t \leq 1} |A_{n, x, y}(s, t) - W_{x, y}(s, t)| = o_p(1), \]

where the covariance structure of \( W_{x, y}(s, t) \) is given by

\[ E[W_{x, y}(s_1, t_1)W_{x, y}(s_2, t_2)] = (|s_1| \wedge |s_2|)I(s_1s_2 > 0)l_1(x, y) + (|t_1| \wedge |t_2|)I(t_1t_2 > 0)l_2(x, y). \]  \hspace{1cm} (7)

Proof. Define set \( A_{s, t} = \{ s, \infty \} \times \{ t, \infty \}, \lambda_1(A_{s, t} \cap A_{0, 0}^c) = -(s \wedge 0)l_1(x, y) - (t \wedge 0)l_2(x, y), \lambda_2(A_{s, t}^c \cap A_{0, 0}) = (s \vee 0)l_1(x, y) + (t \vee 0)l_2(x, y), U_i = h^{-1} \{ n/m(1 - F_1(X_i)) - x \} \) and \( V_i = h^{-1} \{ n/m(1 - F_2(Y_i)) - y \} \) \( (i = 1, \ldots, n) \). When \( s < 0, t < 0 \), we have

\[ \frac{n}{mh} P((U_i, V_i) \in A_{s, t} \cap A_{0, 0}^c) = \frac{n}{mh} \{ P((U_i, V_i) \in A_{s, t}) - P((U_i, V_i) \in A_{0, 0}) \} \]

\[ = h^{-1} \{ l(x, y) - l(x + sh, y + th) + O(A(m/n)) \} \]

\[ = \lambda_1(A_{s, t} \cap A_{0, 0}^c) + o(1). \]
Similarly we can show that the above equation holds for other cases and

\[ \frac{n}{m^h} P((U_i, V_i) \in A_{s,t}^c \cap A_{0,0}) = \lambda_2(A_{s,t}^c \cap A_{0,0}) + o(1). \]

Using Theorem 3.1 of Einmahl (1997) we have

\[ \sup_{-1 \leq s,t \leq 1} \left| \sqrt{mh} \left( \frac{1}{mh} \sum_{i=1}^{n} I((U_i, V_i) \in A_{s,t} \cap A_{0,0}^c) - P((U_i, V_i) \in A_{s,t} \cap A_{0,0}^c) \right) \right| = o_p(1) \]

and

\[ \sup_{-1 \leq s,t \leq 1} \left| \sqrt{mh} \left( \frac{1}{mh} \sum_{i=1}^{n} I((U_i, V_i) \in A_{s,t}^c \cap A_{0,0}) - P((U_i, V_i) \in A_{s,t}^c \cap A_{0,0}) \right) - W_1(A_{s,t} \cap A_{0,0}) \right| = o_p(1), \]

where \( W_1 \) and \( W_2 \) are Wiener processes with covariance structures

\[ E\{W_1(A_{s_1,t_1} \cap A_{0,0}^c)W_1(A_{s_2,t_2} \cap A_{0,0}^c)\} = \lambda_1(A_{s_1,t_1} \cap A_{s_2,t_2} \cap A_{0,0}^c) \]

and

\[ E\{W_2(A_{s_1,t_1} \cap A_{0,0}^c)W_2(A_{s_2,t_2} \cap A_{0,0}^c)\} = \lambda_2(A_{s_1,t_1} \cap A_{s_2,t_2} \cap A_{0,0}^c). \]

Since the sets \( A_{s,t} \cap A_{0,0}^c \) and \( A_{s,t}^c \cap A_{0,0} \) are disjoint, we can show that \( W_1(A_{s,t} \cap A_{0,0}^c) \) and \( W_2(A_{s,t}^c \cap A_{0,0}) \) are independent, and

\[ \sup_{-1 \leq s,t \leq 1} \left| \sqrt{mh} \left( \frac{1}{mh} \sum_{i=1}^{n} I((U_i, V_i) \in A_{s,t} \cap A_{0,0}^c) - 1 \right) \right| = o_p(1). \]

Hence the lemma follows from the facts that

\[ A_{n,x,y} = \sqrt{mh} \left( \frac{1}{mh} \sum_{i=1}^{n} I((U_i, V_i) \in A_{s,t} \cap A_{0,0}^c) - 1 \right) + \frac{l(x + sh, y + th) - l(x, y)}{h}, \]

\[ \frac{n}{m^h} \left\{ P((U_i, V_i) \in A_{s,t} \cap A_{0,0}^c) - P((U_i, V_i) \in A_{s,t}^c \cap A_{0,0}) \right\} - \left( \frac{l(x, y) - l(x + sh, y + th)}{h} \right) = O(h^{-1} A(m/n)) = o \left( \frac{1}{\sqrt{mh}} \right) \]

and \( W_{x,y}(s,t) = W_1(A_{s,t} \cap A_{0,0}^c) - W_2(A_{s,t}^c \cap A_{0,0}) \) has the covariance structure (7). \( \square \)

**Lemma 2.** Under the conditions of Theorem 1,

\[ \begin{cases} 
  m^{-1}h^{-4} D_1 \overset{p}{\rightarrow} - \frac{\partial^2}{\partial x \partial y} l(x, y) \int s^2 k(s) \, ds, \\
  D_2 = o_p(mh^4), \\
  m^{-1}h^{-4} D_3 \overset{p}{\rightarrow} - \frac{\partial^2}{\partial x \partial y} l(x, y) \int s^2 k(s) \, ds.
\end{cases} \]
Proof. Set \( w_i^* = w^*(X_i, Y_i, h, m) = k_h(n/m(1 - F_1(X_i)) - x)k_h(n/m(1 - F_2(Y_i)) - y), \)

\[
D_1^* = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_1(X_i)) - x \right]^2 w_i^*,
\]

\[
D_2^* = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_1(X_i)) - x \right]\left[ \frac{n}{m} (1 - F_2(Y_i)) - y \right] w_i^*,
\]

\[
D_3^* = \sum_{i=1}^{n} \left[ \frac{n}{m} (1 - F_2(Y_i)) - y \right]^2 w_i^*.
\]

It is easy to check that

\[
D_1^* = n \int \int \left[ \frac{n}{m} (1 - F_1(s)) - x \right]^2 k_h\left( \frac{n}{m} (1 - F_1(s)) - x \right) k_h\left( \frac{n}{m} (1 - F_2(t)) - y \right) dF_h(s, t)
\]

\[
= n \int \int s^2 h^2 k(s)k(t) dF_h \left( Q_1 \left( 1 - \frac{m}{n} (x + sh) \right), Q_2 \left( 1 - \frac{m}{n} (y + th) \right) \right)
\]

\[
= nh^2 \int \int \left\{ \left[ 1 - F_n \left( Q_1 \left( 1 - \frac{m}{n} (x + sh) \right) \right) \right] + \left[ 1 - F_n \left( Q_2 \left( 1 - \frac{m}{n} (y + th) \right) \right) \right] \right\} \{ 2sk(s)k'(t) + s^2 k(s)k'(t) \} ds dt
\]

\[
= - nh^2 \int \int \left\{ \left[ 1 - F_n \left( Q_1 \left( 1 - \frac{m}{n} (x + hs) \right) \right), Q_2 \left( 1 - \frac{m}{n} (y + th) \right) \right] \right\} \{ 2sk(s)k'(t) + s^2 k(s)k'(t) \} ds dt
\]

\[
= - nh^2 \int \int \left\{ \left[ 1 - F_n \left( Q_1 \left( 1 - \frac{m}{n} x \right), Q_2 \left( 1 - \frac{m}{n} y \right) \right) \right] \right\} \{ 2sk(s)k'(t) + s^2 k(s)k'(t) \} ds dt
\]

\[
= - mh^3 \int \int \left\{ \frac{l(x + sh, y + th) - l(x, y)}{h} + O_p \left( \frac{1}{\sqrt{mh}} \right) \right\} \{ 2sk(s)k'(t) + s^2 k(s)k'(t) \} ds dt
\]

\[
= - mh^3 \int \int \left\{ sh \frac{\partial^2}{\partial x \partial y} l(x, y) + O(h^2) + O_p \left( \frac{1}{\sqrt{mh}} \right) \right\} \{ 2sk(s)k'(t) + s^2 k(s)k'(t) \} ds dt
\]

\[
= - \left( \int s^2 k(s) ds \right) \frac{\partial^2}{\partial x \partial y} l(x, y)mh^4(1 + o_p(1)). \tag{8}
\]

Similarly we can show

\[
D_2^* = o_p(mh^4) \quad \text{and} \quad D_3^* = - \left( \int s^2 k(s) ds \right) \frac{\partial^2}{\partial x \partial y} l(x, y)mh^4(1 + o_p(1)). \tag{9}
\]

Since

\[
\begin{align*}
\sup_{-\infty < s < \infty} \left| \frac{n}{m} [ 1 - F_{n1}(s) ] - \frac{n}{m} [ 1 - F_1(s) ] \right| & \leq \frac{n}{m} (1 - F_1(s)) - x \leq h = O_p(1/\sqrt{m}), \\
\sup_{-\infty < t < \infty} \left| \frac{n}{m} [ 1 - F_{n2}(t) ] - \frac{n}{m} [ 1 - F_2(t) ] \right| & \leq \frac{n}{m} (1 - F_2(t)) - y \leq h = O_p(1/\sqrt{m}), \\
\sup_{-\infty < s < \infty} \left| \frac{n}{m} [ 1 - F_{n1}(s) ] - \frac{n}{m} [ 1 - F_1(s) ] \right| & \leq \frac{n}{m} (1 - F_1(s)) - y \leq h = O_p(1/\sqrt{m}), \\
\sup_{-\infty < t < \infty} \left| \frac{n}{m} [ 1 - F_{n2}(t) ] - \frac{n}{m} [ 1 - F_2(t) ] \right| & \leq \frac{n}{m} (1 - F_2(t)) - y \leq h = O_p(1/\sqrt{m}).
\end{align*} \tag{10}
\]
for fixed $x > 0$, $y > 0$, we have

$$D_i - D_i^* = o_p(mh^4) \quad \text{for } i = 1, 2, 3. \quad \square$$

(11)

Hence, the lemma follows from (8), (9) and (11).

**Lemma 3.** Under the conditions of Theorem 1,

$$
\begin{align*}
\delta_1 &= -h^3 \sqrt{m(h(1 + o_p(1))) \frac{\partial^2}{\partial x \partial y} l(x, y) \iint W_{x,y}(s,t)s k(s)k(t) \, ds \, dt}, \\
\delta_2 &= -h^3 \sqrt{m(h(1 + o_p(1))) \frac{\partial^2}{\partial x \partial y} l(x, y) \iint W_{x,y}(s,t)t k(s)k(t) \, ds \, dt}, \\
\delta_3 &= \frac{1}{6} h^6 \frac{\partial^2}{\partial x \partial y} \frac{\partial^3}{\partial x^3} l(x, y) \iint s^4 k(s) \, ds + \frac{1}{2} h^6 \frac{\partial^2}{\partial x \partial y} l(x, y) \frac{\partial^3}{\partial x^2 \partial y} \{ \iint s^2 k(s) \, ds \}^2 + o_p(h^6), \\
\delta_4 &= \frac{1}{6} h^6 \frac{\partial^2}{\partial x \partial y} \frac{\partial^3}{\partial x^3} l(x, y) \iint s^4 k(s) \, ds + \frac{1}{2} h^6 \frac{\partial^2}{\partial x \partial y} l(x, y) \frac{\partial^3}{\partial x^2 \partial y} \{ \iint s^2 k(s) \, ds \}^2 + o_p(h^6).
\end{align*}
$$

**Proof.** Set

$$
\delta_1^* = \sum_{i=1}^n \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \frac{n}{m} \left[ 1 - F_n \left( \frac{Q_1(1 - \frac{m}{n} x), Q_2(1 - \frac{m}{n} y)}{1} \right) \right] \\
- l \left( \frac{n}{m} [1 - F_1(X_i)], \frac{n}{m} [1 - F_2(Y_i)] \right) \left[ \frac{n}{m} (1 - F_1(X_i)) - x \right] w_i^*,
\right.
$$

$$
\delta_2^* = \sum_{i=1}^n \left\{ \frac{n}{m} [1 - F_n(X_i, Y_i)] - \frac{n}{m} \left[ 1 - F_n \left( \frac{Q_1(1 - \frac{m}{n} x), Q_2(1 - \frac{m}{n} y)}{1} \right) \right] \\
- l \left( \frac{n}{m} [1 - F_1(X_i)], \frac{n}{m} [1 - F_2(Y_i)] \right) \left[ \frac{n}{m} (1 - F_2(Y_i)) - y \right] w_i^*.
\right.
$$

$u(X_i) = [n/m(1 - F_1(X_i)) - x]h^{-1}$ and $v(Y_i) = [n/m(1 - F_2(Y_i)) - y]h^{-1}$. Then

$$
F_1(X_i) = 1 - \frac{m}{n} (x + hu(X_i)) \quad \text{and} \quad F_2(Y_i) = 1 - \frac{m}{n} (y + hv(Y_i)).
$$

Note that for any $X_j$, we have with probability one

$$I(X_j \leq X_i) = I(F_1(X_j) \leq F_1(X_i))$$

$$= I \left( F_1(X_j) < 1 - \frac{m}{n} (x + hu(X_i)) \right)$$

$$= \lim_{\varepsilon \rightarrow 0} I \left( F_1(X_j) < 1 - \frac{m}{n} (x + hu(X_i)) + \varepsilon \right)$$

$$= \lim_{\varepsilon \rightarrow 0} I \left( X_j < Q_1 \left( 1 - \frac{m}{n} (x + hu(X_i)) + \varepsilon \right) \right)$$

$$= I \left( X_j < Q_1 \left( 1 - \frac{m}{n} (x + hu(X_i)) + \varepsilon \right) \right),$$

where $Q_1(s+) = \lim_{\varepsilon \rightarrow 0} Q_1(s + \varepsilon)$ is the right limit of $Q_1$. In the same manner we have with probability one

$$I(Y_j \leq Y_i) = I \left( X_j \leq Q_2 \left( 1 - \frac{m}{n} (y + hv(Y_i)) + \varepsilon \right) \right),$$
and thus

$$\delta_i^* = -\sum_{i=1}^{n} \left\{ \frac{n}{m} \left[ F_n \left( Q_1 \left( 1 - \frac{m}{n} (x + hu(X_i)) \right), Q_2 \left( 1 - \frac{m}{n} (y + hv(Y_i)) \right) \right) \right. \right.$$

$$- F_n \left( Q_1 \left( 1 - \frac{m}{n} x \right), Q_2 \left( 1 - \frac{m}{n} y \right) \right) \left. \right] + l(x + u(X_i)h, y + v(Y_i)h) - l(x, y) \right\}$$

$$\times u(X_i)h k_h(u(X_i)) k_h(v(Y_i))$$

$$= - \frac{h^2}{\sqrt{mh}} \sum_{i=1}^{n} A_{n,x,y}(u(X_i) - v(Y_i) - u(X_i)k(u(X_i))k(v(Y_i)))$$

where $A_{n,x,y}(s - t) = \lim_{\varepsilon \to 0} A_{n,x,y}(s + \varepsilon, t + \varepsilon)$. Notice that any summand in the above equation is zero if $|u(X_i)| \geq 1$ or $|v(Y_i)| \geq 1$. Since $W_{x,y}(s, t)$ is continuous in $s$ and $t$, it follows from Lemma 1 that

$$\delta_1^* = - \frac{h^2}{\sqrt{mh}} (1 + o_p(1)) \sum_{i=1}^{n} W_{x,y}(u(X_i) - v(Y_i) - u(X_i)k(u(X_i))k(v(Y_i)))$$

Similarly,

$$\delta_2^* = - \frac{h^2}{\sqrt{mh}} (1 + o_p(1)) \int \int W_{x,y}(u(s), v(t)) u(s) k(u(s)) k(v(t)) \, dF_n(s, t)$$

By (10), we have

$$\delta_i - \delta_i^* = o_p \left( \frac{h^3 \sqrt{mh}}{\sqrt{m}} \right) \text{ for } i = 1, 2.$$
Hence the equations for $\delta_1$ and $\delta_2$ stated in Lemma 3 follow from (12)–(14). Furthermore, the equations for $\delta_3$ and $\delta_4$ stated in Lemma 3 follow from applying Taylor’s expansion to $l(n/m[1 - F_{n1}(X_i)], n/m[1 - F_{n2}(Y_i)])$. □

**Proof of Theorem 1.** Since

$$d_1 - D_1 l_1(x, y) - D_2 l_2(x, y) = \delta_1 + \delta_3$$

and

$$d_2 - D_2 l_1(x, y) - D_3 l_2(x, y) = \delta_2 + \delta_4,$$

we have

$$\hat{l}_1(x, y) - l_1(x, y) = \frac{[d_1 - D_1 l_1(x, y) - D_2 l_2(x, y)]D_3 - [d_2 - D_2 l_1(x, y) - D_3 l_2(x, y)]D_2}{D_1 D_3 - D_2^2}$$

$$= \frac{1}{1 - D_2^2/(D_1 D_3)} \frac{\delta_1 + \delta_3}{D_1} - \frac{D_2/D_1}{1 - D_2^2/(D_1 D_3)} \frac{\delta_2 + \delta_4}{D_3}$$

and

$$\hat{l}_2(x, y) - l_2(x, y) = \frac{[d_2 - D_2 l_1(x, y) - D_3 l_2(x, y)]D_1 - [d_1 - D_1 l_1(x, y) - D_2 l_2(x, y)]D_2}{D_1 D_3 - D_2^2}$$

$$= \frac{1}{1 - D_2^2/(D_1 D_3)} \frac{\delta_2 + \delta_4}{D_2} - \frac{D_2/D_3}{1 - D_2^2/(D_1 D_3)} \frac{\delta_1 + \delta_3}{D_1}.$$ 

Hence, by Lemmas 2 and 3,

$$\sqrt{m h} \left\{ \hat{l}_1(x, y) - l_1(x, y) + h^2 \left( \frac{1}{6} (\bar{\partial}^3 / \partial x^3) l(x, y) \int s^4 k(s) ds + \frac{1}{2} (\bar{\partial}^3 / \partial x \partial y) l(x, y) (\int s^2 k(s) ds)^2 \right) \right\}$$

$$= \frac{\iint W_{x,y}(s,t) s k(s) k(t) ds dt}{\int s^2 k(s) ds} + o_p(1)$$

and

$$\sqrt{m h} \left\{ \hat{l}_2(x, y) - l_2(x, y) + h^2 \left( \frac{1}{6} (\bar{\partial}^3 / \partial y^3) l(x, y) \int s^4 k(s) ds + \frac{1}{2} (\bar{\partial}^3 / \partial x \partial y) l(x, y) (\int s^2 k(s) ds)^2 \right) \right\}$$

$$= \frac{\iint W_{x,y}(s,t) t k(s) k(t) ds dt}{\int s^2 k(s) ds} + o_p(1).$$

Therefore, Theorem 1 follows from that

$$E \left\{ \iint W_{x,y}(s,t) s k(s) k(t) ds dt \right\}^2 = 2l_1(x, y) \int_0^1 \int_0^1 (s \wedge t) k(s) k(t) ds dt$$

$$+ 2l_2(x, y) \int_0^1 \int_0^1 (s \wedge t) s t k(s) k(t) ds dt,$$

$$E \left\{ \iint W_{x,y}(s,t) t k(s) k(t) ds dt \right\}^2 = 2l_1(x, y) \int_0^1 \int_0^1 (s \wedge t) s t k(s) k(t) ds dt$$

$$+ 2l_2(x, y) \int_0^1 \int_0^1 (s \wedge t) k(s) k(t) ds dt$$

and

$$E \left\{ \iint W_{x,y}(s,t) s k(s) k(t) ds dt \right\} \left\{ \iint W_{x,y}(s,t) t k(s) k(t) ds dt \right\} = 0. \quad \square$$
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