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Maximum likelihood estimation of extreme value index for irregular cases

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ABSTRACT

A method in analyzing extremes is to fit a generalized Pareto distribution to the exceedances over a high threshold. By varying the threshold according to the sample size [Smith, R.L., 1987. Estimating tails of probability distributions. Ann. Statist. 15, 1174–1207] and [Drees, H., Ferreira, A., de Haan, L., 2004. On maximum likelihood estimation of the extreme value index. Ann. Appl. Probab. 14, 1179–1201] derived the asymptotic properties of the maximum likelihood estimates (MLE) when the extreme value index is larger than $-\frac{1}{2}$. Recently Zhou [2009. Existence and consistency of the maximum likelihood estimator for the extreme value index. J. Multivariate Anal. 100, 794–815] showed that the MLE is consistent when the extreme value index is larger than -1. In this paper, we study the asymptotic distributions of MLE when the extreme value index is in between -1 and $-\frac{1}{2}$ (including $-\frac{1}{2}$). Particularly, we consider the MLE for the endpoint of the generalized Pareto distribution and the extreme value index and show that the asymptotic limit for the endpoint estimate is non-normal, which connects with the results in Woodroofe [1974. Maximum likelihood estimation of translation parameter of truncated distribution II. Ann. Statist. 2, 474–488]. Moreover, we show that same results hold for estimating the endpoint of the underlying distribution, which generalize the results in Hall [1982. On estimating the endpoint of a distribution. Ann. Statist. 10, 556–568] to irregular case, and results in Woodroofe [1974. Maximum likelihood estimation of translation parameter of truncated distribution II. Ann. Statist. 2, 474–488] to the case of unknown extreme value index.

1. Introduction

Let $X, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with a distribution function (d.f.) F, which lies in the domain of attraction of an extreme value distribution, i.e., there exist $a_n > 0$ and $b_n \in R$ such that

$$\lim_{n\to\infty} P\left(\frac{\max_{1\leqslant i\leqslant n} X_i - b_n}{a_n} \leqslant x\right) = G(x) \tag{1.1}$$

for all continuous points of G, where G is a non-degenerate distribution. Based on (1.1), one can infer the tail properties of F, and hence extrapolate data into a far tail region of F; see Chapter 4 of De Haan and Ferreira (2006). It is known that G in (1.1) can be written as

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}$$

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for some $\gamma \in R$ and $1+\gamma x>0$, and condition (1.1) implies that there exists a positive non-decreasing function σ such that

$$\lim_{t\to\theta} P\left(\frac{X-t}{\sigma(t)}>x|X>t\right)=(1+\gamma x)^{-1/\gamma}$$

for all $x>0$ and $1+\gamma x>0$, where $\theta=\sup\{x: F(x)<1\}$ is the right endpoint of F. Therefore, given a fixed high threshold u, the conditional distribution of the exceedance above the threshold, i.e. $P(X-u\leqslant x|X>u)$, can be approximated by the so-called generalized Pareto distribution

$$H(x)=1-(1+\gamma x/\sigma(u))^{-1/\gamma}, \tag{1.2}$$

see Chapter 3.1 of De Haan and Ferreira (2006).

A parametric way in analyzing extremes is to fit the generalized Pareto distribution (1.2) to the exceedances over a fixed high threshold. With this setup, it follows from Smith (1985) that the rate of convergence of the maximum likelihood estimates (MLE) for parameters γ and $\tau := -\sigma(u)/\gamma$ depends on whether $\gamma>-\frac{1}{2}$, $\gamma=-\frac{1}{2}$ or $-1<\gamma<-\frac{1}{2}$. Note that the parameter τ corresponds to the endpoint of H when $\gamma<0$, and the setting in Smith (1985) is broader than the assumption of generalized Pareto distribution.

A difficulty for this parametric approach is how to quantify the influence of the fixed high threshold. An alternative way in analyzing extremes is to start from condition (1.1) directly, which is considered as a semiparametric model. Under this setup, one may choose a sequence of thresholds $u=u_n$ and estimate γ and $\sigma(u_n)$ by maximum likelihood estimation. For example, Smith (1987) derived the limit of MLE for (γ,σ) when $\gamma\geqslant-\frac{1}{2}$, but did not obtain the limit for the case $-1<\gamma<-\frac{1}{2}$. Instead, Smith (1987) considered a different estimator from MLE for the case $-1<\gamma<-\frac{1}{2}$. Recently, Drees et al. (2004) revisited the maximum likelihood estimate of (γ,σ) by taking the threshold as the $(k+1)$-th largest order statistic, and derived the limit when $\gamma>-\frac{1}{2}$ by employing a weighted approximation of tail quantile processes. Under condition (2.1) given below, Zhou (2009) showed that the MLE of (γ,σ) exists and is consistent when $\gamma>-1$. Therefore, a natural question is what the limit and the rate of convergence of MLE are when $\gamma\in(-1,-\frac{1}{2}]$. Answering this question completes the results in Smith (1987) and Drees et al. (2004).

It is known that condition (1.1) with $\gamma<0$ implies that the right endpoint of F is finite (i.e. $\theta<\infty$) and

$$\lim_{t\to\infty}\frac{1-F(\theta-(tx)^{-1})}{1-F(\theta-t^{-1})}=x^{1/\gamma}\quad\text{for } x>0. \tag{1.3}$$

Based on (1.3), the endpoint θ can be estimated; see Hall (1982), Loh (1984), Falk (1995), Athreya and Fukuchi (1997) and Hall and Wang (1999) for the regular case $\gamma\in(-\frac{1}{2},0)$, and Woodroofe (1974) for the irregular case $\gamma\in(-1,-\frac{1}{2})$. Note that Woodroofe (1974) assumed that γ is known. In this paper, we focus on the irregular case as Woodroofe (1974), but take the same setup as Drees et al. (2004) without assuming that γ is unknown. That is, we assume (1.1) holds for some $\gamma\in(-1,-\frac{1}{2}]$. Therefore, we have one endpoint for the generalized Pareto distribution (1.2) and another endpoint for the underlying distribution F. From Woodroofe (1974), one may conjecture that the maximum likelihood estimators for both endpoints have a faster rate of convergence compared to the MLE for γ, and the corresponding limit laws for both endpoints are non-normal.

In this paper, this conjecture is proved when the MLE of τ is normalized by a sequence of random variables. It is also shown that the MLE of τ, normalized by a sequence of non-random constants, has a normal limit with a slower convergence rate. This phenomenon can be explained by the fact that we employ a stochastic threshold in the maximum likelihood estimation procedure and thus the true value of τ is essentially related to the random threshold. When considering the maximum likelihood estimate for (γ,σ), the rates of convergence for these two parameters turn out to be of the same order and the limits are both normal distribution. This is different from the limit for (γ,τ). At first glance, the difference is mysterious: τ is a function of γ and σ and studying MLE for (γ,σ) is equivalent to that for (γ,τ). However, we should keep in mind that the case $\gamma\in(-1,-\frac{1}{2}]$ is irregular and so the invariant property of MLE does not hold. In fact, since $\hat\sigma=-\hat\gamma\hat\tau$, the convergence rate of $\hat\sigma$ to σ is determined by that of $\hat\gamma$ to γ.

Based on the above study on estimating τ, we also derive the joint asymptotic limit of estimators for γ and the endpoint θ in (1.3), which shows that the limit of the endpoint estimator for F is non-normal. Since Li and Peng (2008) pointed out that maximum likelihood estimation for γ and θ based on the generalized Pareto distribution is the same as that based on model (1.3), our results extend the results in Hall (1982) to irregular case and generalize the results in Woodroofe (1974) to the case of jointly estimating the extreme value index and endpoint with an unknown extreme value index.

We organize this paper as follows. The main results are given in Section 2. We summarize our conclusions in Section 3. All proofs are given in Appendix A.

2. Main results

Let $X_{n,1}\leqslant\cdots\leqslant X_{n,n}$ be the order statistics of the i.i.d. random variables $X_1,\ldots,X_n$. Let $k=k_n$ be an intermediate sequence of integers satisfying

$$k\to\infty\quad\text{and}\quad\frac{k}{n}\to 0\ \text{ as } n\to\infty. \tag{2.1}$$

Given $X_{n,n-k}$, the conditional likelihood function of

$$(X_{n,n-k+1}-X_{n,n-k},\ldots,X_{n,n}-X_{n,n-k})$$

can be approximated by

$$L(\gamma,\sigma)=\prod_{i=1}^{k}h(X_{n,n-i+1}-X_{n,n-k};\gamma,\sigma),$$

where $h(x;\gamma,\sigma)=(\partial/\partial x)H(x)$ is the density function of the generalized Pareto distribution $H(x)$ defined in (1.2). It follows from Drees et al. (2004) that the score equations are

$$\frac{1}{k}\sum_{i=1}^{k}\log\{1-(X_{n,n-i+1}-X_{n,n-k})/\tau\}=\gamma, \tag{2.2}$$

$$\frac{1}{k}\sum_{i=1}^{k}\frac{1}{1-(X_{n,n-i+1}-X_{n,n-k})/\tau}=\frac{1}{1+\gamma} \tag{2.3}$$

and

$$1-(X_{n,n-i+1}-X_{n,n-k})/\tau>0 \quad \text{for all } i=1,\ldots,k \tag{2.4}$$

when $\gamma\neq 0$ and $\tau=-\sigma/\gamma$. Note that Eq. (2.4) is not used in Drees et al. (2004) because the solution of (2.2) and (2.3) automatically fulfills this requirement when $\gamma>-\frac{1}{2}$. But this constraint is necessary in determining the limit of MLE when $-1<\gamma\leqslant-\frac{1}{2}$; see Theorem 1.

For deriving the asymptotic limit of the MLE, similar to Drees et al. (2004), we need a second order condition: suppose there exist two real functions $a(t)$ and $A(t)\to 0$ (as $t\to\infty$) such that

$$\lim_{t\to\infty}\frac{(U(tx)-U(t))/a(t)-(x^{\gamma}-1)/\gamma}{A(t)}=\Psi(x):=\int_1^x s^{\gamma-1}\int_1^s u^{\rho-1}\,du\,ds \tag{2.5}$$

for some $\gamma>-1$ and some $\rho\leqslant 0$, where $U(t)$ denotes the left-continuous inverse of $1/(1-F(t))$. It follows from Theorem 3.3.5 of De Haan and Ferreira (2006) that

$$\sigma(t)=a(U^{-}(t)), \tag{2.6}$$

where U^{-} denotes the left-continuous inverse function of U.

Throughout, we assume $-1<\gamma\leqslant-\frac{1}{2}$ and let $(\hat\gamma,\hat\tau)$ denote the solution of (γ,τ) to Eqs. (2.2)–(2.4). Denote the true value of γ by γ_0. In virtue of the random threshold $X_{n,n-k}$, we will normalize the MLE of τ by $a(U^{-}(X_{n,n-k}))$. It follows from (2.5) that $a(t)$ has a positive sign for large t, and hence (2.4) is equivalent to

$$\tau/a(U^{-}(X_{n,n-k}))>\{X_{n,n}-X_{n,n-k}\}/a(U^{-}(X_{n,n-k})):=M_n. \tag{2.7}$$

Put

$$f_n(t)=\frac{1}{k}\sum_{i=1}^{k}\log\left\{1-\frac{X_{n,n-i+1}-X_{n,n-k}}{ta(U^{-}(X_{n,n-k}))}\right\}+1,$$

$$g_n(t)=\frac{1}{k}\sum_{i=1}^{k}\left\{1-\frac{X_{n,n-i+1}-X_{n,n-k}}{ta(U^{-}(X_{n,n-k}))}\right\}^{-1}$$

and

$$h_n(t)=g_n(t)-\frac{1}{f_n(t)}.$$

Put $\delta_n=|A(n/k)\log(k^{-\gamma_0}|A(n/k)|)|$. First we show that there exits τ such that $\tau/a(U^{-}(X_{n,n-k}))$ lies between $M_n(1+\delta_n)$ and $-1/\gamma_0+\varepsilon$ for some $\varepsilon>0$ as defined in the following proposition.

Proposition 1. *Under conditions* (2.1), (2.5), $\gamma\in(-1,-\frac{1}{2}]$ *and*

$$k^{-\gamma_0}A(n/k)\to 0,\quad k^{-1}|\log|A(n/k)||\to 0 \quad \text{as } n\to\infty, \tag{2.8}$$

we have

$$\lim_{n\to\infty} P(h_n(M_n(1+\delta_n))>0)=1, \tag{2.9}$$

$$\lim_{n\to\infty} P(h_n(-1/\gamma_0+\varepsilon)<0)=1 \tag{2.10}$$

and

$$\lim_{n\to\infty} P(h_n'(t)<0 \quad \text{for all } t\in[M_n(1+\delta_n),-1/\gamma_0+\varepsilon])=1, \tag{2.11}$$

where $\varepsilon>0$ *is small enough.*

It follows from Proposition 1 that, with a probability tending to one, there exits a unique solution, say $(\hat\gamma,\hat\tau)$, which satisfies that

$$\begin{cases} \hat\tau>(X_{n,n}-X_{n,n-k})(1+\delta_n),\\ h_n(\hat\tau/a(U^{-}(X_{n,n-k})))=0,\\ \hat\gamma=\dfrac{1}{k}\sum\limits_{i=1}^{k}\log\{1-(X_{n,n-i+1}-X_{n,n-k})/\hat\tau\}. \end{cases} \tag{2.12}$$

Note that there may exist a $\tau\in(X_{n,n}-X_{n,n-k},(X_{n,n}-X_{n,n-k})(1+\delta_n))$ such that $h_n(\tau/a(U^{-}(X_{n,n-k})))=0$. In this case, it is obvious that this solution has the same asymptotic limit as M_n. From now on, we focus on the study on $(\hat\gamma,\hat\tau)$ satisfying (2.12).

Theorem 1. *Assume* $\gamma_0\in(-1,-\frac{1}{2})$. *Under conditions* (2.1), (2.5) and (2.8) *we have*

$$(\sqrt{k}\{\hat\gamma-\gamma_0\},k^{-\gamma_0}\{\hat\tau/a(U^{-}(X_{n,n-k}))+1/\gamma_0\})\xrightarrow{d}(W,V),$$

where W *and* V *are independent random variables,* $W\sim \mathrm{N}(0,\gamma_0^2)$, *and the distribution of* V *is given by*

$$V(y):=P(V\leqslant y)=\begin{cases} P(W_y\leqslant 0) & \text{if } y\geqslant 0,\\ P(W_y\leqslant 0)\exp\{-(\gamma_0 y)^{-1/\gamma_0}\} & \text{if } y<0, \end{cases}$$

where W_y *is defined in Lemma* 2.

Theorem 2. *Assume* $\gamma_0=-\frac{1}{2}$. *If conditions* (2.1), (2.8) and (2.5) *hold, then we have*

$$(\sqrt{k}\{\hat\gamma-\gamma_0\},(k\ln k)^{-\gamma_0}\{\hat\tau/a(U^{-}(X_{n,n-k}))+1/\gamma_0\})\xrightarrow{d}(W,V'),$$

where W *and* V' *are independent random variables,* $W\sim\mathrm{N}(0,\gamma_0^2)$, *and* $V'\sim\mathrm{N}(0,\gamma_0^{-2})$.

Remark 1. Note that the case $\gamma_0=-\frac{1}{2}$ is not studied in Woodroofe (1974).

Note that in Theorems 1 and 2, we can treat the MLE of τ as an estimate of $-a(U^{-}(X_{n,n-k}))/\gamma_0$, which is the random endpoint of the generalized Pareto distribution H. When studying $\hat\sigma$, one may take $a(n/k)$ as the true value as in Drees et al. (2004). In this case, we compare $\hat\tau$ with the non-random endpoint $-a(n/k)/\gamma_0$. By writing

$$\hat\tau/a(n/k)+1/\gamma_0=\{\hat\tau/a(U^{-}(X_{n,n-k}))+1/\gamma_0\}\frac{a(U^{-}(X_{n,n-k}))}{a(n/k)}+\gamma_0^{-1}\left\{\frac{a(U^{-}(X_{n,n-k}))}{a(n/k)}-1\right\}, \tag{2.13}$$

the following theorem shows that the rate of convergence is $k^{-1/2}$, which is slower than k^{γ_0} for $\gamma_0\in(-1,-\frac{1}{2})$ or $(k\ln k)^{\gamma_0}$ for $\gamma_0=-\frac{1}{2}$.

Theorem 3. *Let* $-1<\gamma_0\leqslant-\frac{1}{2}$. *Assume conditions* (2.1), (2.5) and (2.8) *hold. Then we have as* $n\to\infty$

$$(\sqrt{k}\{\hat\gamma-\gamma_0\},\sqrt{k}\{\hat\tau/a(n/k)+1/\gamma_0\})\xrightarrow{d}(W,W^*),$$

where W *and* W^* *are independent random variables,* $W\sim\mathrm{N}(0,\gamma_0^2)$ *and* $W^*\sim\mathrm{N}(0,1)$.

An alternative interest is to estimate σ instead of the endpoint τ. In this case, the MLE for σ is $\hat\sigma=-\hat\gamma\hat\tau$ and its asymptotic limit is given as follows. Here we compare the limits of $\hat\sigma$ after normalized by both the non-random constant $a(n/k)$ and the random variable $a(U^{-}(X_{n,n-k}))$, respectively.

Theorem 4. *Assume all conditions in Theorem* 3 *are satisfied. Then we have*

$$(\sqrt{k}\{\hat{\gamma}-\gamma_0\},\sqrt{k}\{\hat{\sigma}/a(n/k)-1\},\sqrt{k}\{\hat{\sigma}/a(U^{-}(X_{n,n-k}))-1\})\xrightarrow{d}(W,\gamma_0 W^*+W/\gamma_0,W/\gamma_0),$$

where W *and* W^* *are given in Theorem* 3.

Next we estimate the endpoint θ of F. Based on $\hat{\gamma}$ and $\hat{\tau}$, one can estimate θ by $\hat{\theta}=X_{n,n-k}+\hat{\tau}$. As pointed out in Li and Peng (2008), estimators $\hat{\gamma}$ and $\hat{\theta}$ are the same as the maximum likelihood estimator in Hall (1982). Hence the following results extend Hall (1982) to irregular case.

Theorem 5. *Assume* $\gamma_0\in(-1,-\frac{1}{2})$. *Under conditions* (2.1), (2.5) and (2.8), *we have*

$$\left(\sqrt{k}\{\hat{\gamma}-\gamma_0\},k^{-\gamma_0}\frac{\hat{\theta}-\theta}{a(n/k)}\right)\xrightarrow{d}(W,V),$$

where (W,V) *is defined in Theorem* 1.

Theorem 6. *Assume* $\gamma_0=-\frac{1}{2}$. *If conditions* (2.1), (2.8) and (2.5) *hold, then we have*

$$\left(\sqrt{k}\{\hat{\gamma}-\gamma_0\},(k\ln k)^{-\gamma_0}\frac{\hat{\theta}-\theta}{a(n/k)}\right)\xrightarrow{d}(W,V'),$$

where (W,V') *is defined in Theorem* 2.

Remark 2. Write

$$\frac{k^{-\gamma_0}}{a(n/k)}a(n)-1=k^{-\gamma_0}A(n/k)\frac{\dfrac{a\left(\frac{n}{k}k\right)}{a\left(\frac{n}{k}\right)}-k^{\gamma_0}}{A(n/k)}.$$

It follows from (2.8) and Theorem 2.3.6 of De Haan and Ferreira (2006) that $k^{-\gamma_0}a(n)/a(n/k)\to 1$, which implies that $\hat{\theta}$ has the same rate of convergence as $X_{n,n}$ when $\gamma\in(-1,-\frac{1}{2})$. But it is true that $\theta-\hat{\theta}\leqslant\theta-(\{X_{n,n-k}+(X_{n,n}-X_{n,n-k})\}=\theta-X_{n,n}$, i.e., $\hat{\theta}$ is always closer to the true value than $X_{n,n}$. When $\gamma_0=-\frac{1}{2}$, $\hat{\theta}$ has a faster rate of convergence than $X_{n,n}$.

Remark 3. It is known that the endpoint estimators in Dekkers and de Haan (1989) and Dekkers et al. (1989) have the rate of convergence $k^{-1/2}a(n/k)$ when $\lim_{n\to\infty}\sqrt{k}A(n/k)\in(-\infty,\infty)$. Hence, $\hat{\theta}$ is better than the endpoint estimators in Dekkers and de Haan (1989) and Dekkers et al. (1989) when $\gamma\in(-1,-\frac{1}{2})$. However, the estimators for γ in Dekkers and de Haan (1989) and Dekkers et al. (1989) have a faster rate of convergence than the maximum likelihood estimator when $\gamma\in(-1,-\frac{1}{2})$ since (2.8) is required in our Theorem 5.

3. Conclusions

We generalize the study in Drees et al. (2004) to the irregular case $\gamma\in(-1,-\frac{1}{2}]$, hence the maximum likelihood estimator for a generalized Pareto distribution becomes known for $\gamma>-1$. As pointed out in Li and Peng (2008), when $\gamma<0$, models (1.2) and (1.3) are equivalent in the sense that the maximum likelihood estimators for the extreme value index and the endpoint of underlying distribution are the same. Therefore, although we study model (1.2), our results generalize the study in Hall (1982) from the regular case $\gamma\in(-\frac{1}{2},0)$ to the irregular case $\gamma\in(-1,-\frac{1}{2}]$. Also our results extend those in Woodroofe (1974) from known γ to unknown γ.

It is known that maximum for the endpoint has a better rate than the endpoint estimators in Dekkers and de Haan (1989) and Dekkers et al. (1989) when $\gamma<-\frac{1}{2}$, but it has a worse rate when $\gamma\geqslant-\frac{1}{2}$. Our results, combining with those in Drees et al. (2004) show that the maximum likelihood estimator for the endpoint always achieves the best rate of convergence for $\gamma>-1$.

Appendix A. Proofs

Let $Y_1,\dots,Y_n$ be i.i.d. random variables with distribution $1-1/x$ and $Y_{n,1}\leqslant\cdots\leqslant Y_{n,n}$ denote the order statistics of $Y_1,\dots,Y_n$. Consider another independent sequence of i.i.d. random variables $Y_1^*,\dots,Y_k^*$ with distribution function $1-1/x$. Denote $Y_{k,1}^*\leqslant\cdots\leqslant Y_{k,k}^*$

as their order statistics. It is well known that

$$\{Y_{n,n-i+1}/Y_{n,n-k}\}_{i=1}^{k} \stackrel{\mathrm{d}}{=} \{Y_{k,k-i+1}^{*}\}_{i=1}^{k}. \tag{A.1}$$

The following lemma comes from Lemma 5.2 of Draisma et al. (1999).

Lemma 1. *Let f be a measurable function. Suppose there exist a real parameter α and functions $a_1(t)>0$ and $A_1(t)\to 0$ such that for all $x>0$*

$$\lim_{t\to\infty}\frac{\dfrac{f(tx)-f(t)}{a_1(t)}-\dfrac{x^\alpha-1}{\alpha}}{A_1(t)}=H_1(x)=\frac{1}{\beta}\left\{\frac{x^{\alpha+\beta}-1}{\alpha+\beta}-\frac{x^\alpha-1}{\alpha}\right\},$$

where $\beta\leqslant 0$. Then for any $\varepsilon>0$ there exists $t_0>0$ such that for all $t\geqslant t_0, tx\geqslant t_0$,

$$\left|\frac{\dfrac{f(tx)-f(t)}{a_1(t)}-\dfrac{x^\alpha-1}{\alpha}}{A_1(t)}-H_1(x)\right|\leqslant\varepsilon\{1+x^\alpha+2x^{\alpha+\beta}e^{\varepsilon|\log x|}\}.$$

Put

$$l_1(x;t)=\log\left\{1-\frac{x^{\gamma_0}-1}{t\gamma_0}\right\}-\gamma_0 \quad\text{and}\quad l_2(x;t)=\left\{1-\frac{x^{\gamma_0}-1}{t\gamma_0}\right\}^{-1}-\frac{1}{1+\gamma_0}.$$

In Lemmas 2 and 3 and their proofs below, we use e^{ix} to denote complex number $\cos x+i\sin x$. In the two lemmas, we only assume $k\to\infty$. One can assume that $k=k_n$ depends on n like condition (2.1) but for all random quantities of interest in these two lemmas, their distributions depend on n through k_n only.

Lemma 2. *Assume $\gamma_0\in(-1,-\frac{1}{2})$.*

(i) *For $t\geqslant 0$, we have*

$$\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}l_1(Y_j^*;-1/\gamma_0+tk^{\gamma_0}),\ k^{\gamma_0}\sum_{j=1}^{k}l_2(Y_j^*;-1/\gamma_0+tk^{\gamma_0})\right)\stackrel{\mathrm{d}}{\to}(W,W_t) \tag{A.2}$$

as $k\to\infty$, where W and W_t are independent, $W\sim \mathrm{N}(0,\gamma_0^2)$ and the characteristic function of W_t is

$$E\{e^{i\lambda W_t}\}=\exp\left\{\int_0^\infty\left\{\exp\left(i\lambda\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-i\lambda\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+\int_0^\infty i\lambda\frac{t\gamma_0}{x^{\gamma_0}(x^{\gamma_0}-t\gamma_0)}x^{-2}\,dx\right\}.$$

(ii) *For $t<0$, conditional on $(\max_{1\leqslant j\leqslant k}Y_j^*)^{\gamma_0}/\gamma_0<tk^{\gamma_0}$, (A.2) holds with*

$$E\{e^{i\lambda W_t}\}=\exp\left\{\int_0^{a(t,\gamma_0)}\left\{\exp\left(i\lambda\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-i\lambda\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+\int_{a(t,\gamma_0)}^{(t\gamma_0)^{1/\gamma_0}}\left\{\exp\left(i\lambda\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1\right\}x^{-2}\,dx\right\},$$

where $a(t,\gamma_0)\in(0,(t\gamma_0)^{1/\gamma_0})$ is the unique solution of the following equation:

$$\varphi(x):=\int_0^x\frac{t\gamma_0}{y^{\gamma_0}(y^{\gamma_0}-t\gamma_0)}y^{-2}\,dy-\frac{x^{-\gamma_0-1}}{1+\gamma_0}=0. \tag{A.3}$$

(iii) *For $t\geqslant 0$, we have*

$$\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}l_1(Y_j^*;-1/\gamma_0),k^{\gamma_0}\sum_{j=1}^{k}l_2(Y_j^*;-1/\gamma_0+tk^{\gamma_0})\right)\stackrel{\mathrm{d}}{\to}(W,W_t) \tag{A.4}$$

as $k\to\infty$, where W and W_t are given in part (i).

(iv) *For $t<0$, conditional on $(\max_{1\leqslant j\leqslant k}Y_j^*)^{\gamma_0}/\gamma_0<tk^{\gamma_0}$, (A.4) holds with W_t given in part* (ii).

Proof. (i) It is easy to check that

$$\begin{aligned}
\delta_1(t_1) &:= E\{\exp(it_1k^{-1/2}l_1(Y_j^*;-1/\gamma_0+tk^{\gamma_0}))\}-1\\
&= \int_1^\infty \{\exp(it_1k^{-1/2}l_1(x;-1/\gamma_0+tk^{\gamma_0}))-1\}x^{-2}\,dx\\
&= \int_1^\infty \{it_1k^{-1/2}l_1(x;-1/\gamma_0+tk^{\gamma_0})-\frac{1}{2}t_1^2k^{-1}l_1^2(x;-1/\gamma_0+tk^{\gamma_0})\}x^{-2}\,dx+o(k^{-1})\\
&= \int_1^\infty \{it_1k^{-1/2}(\gamma_0\log x-\gamma_0)-\frac{1}{2}t_1^2k^{-1}(\gamma_0\log x-\gamma_0)^2\}x^{-2}\,dx+o(k^{-1})\\
&= -\frac{1}{2}t_1^2\gamma_0^2k^{-1}+o(k^{-1}),
\end{aligned}\tag{A.5}$$

$$\begin{aligned}
\delta_2(t_2) &:= E\{\exp(it_2k^{\gamma_0}l_2(Y_j^*;-1/\gamma_0+tk^{\gamma_0})\}-1\\
&= \int_1^\infty \exp\left\{it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right\}\exp\left\{-it_2\frac{tk^{2\gamma_0}\gamma_0}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right\}\exp\left\{-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right\}x^{-2}\,dx-1\\
&= \int_1^\infty \exp\left\{it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right\}\exp\left\{-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right\}x^{-2}\,dx-1+o(k^{-1})\\
&= \int_1^\infty \exp\left\{it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right\}\left\{1-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right\}x^{-2}\,dx-1+o(k^{-1})\\
&= \int_1^\infty \left\{\exp\left(it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right)-1-it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right\}\left(1-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right)x^{-2}\,dx\\
&\quad+\int_1^\infty \left(1+it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}\right)\left(1-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right)x^{-2}\,dx-1+o(k^{-1})\\
&= k^{-1}\int_{1/k}^\infty \left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}\left(1-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right)x^{-2}\,dx\\
&\quad+\int_1^\infty \left(it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}-it_2\frac{k^{\gamma_0}}{1+\gamma_0}\right)x^{-2}\,dx+o(k^{-1})\\
&= k^{-1}\int_{1/k}^\infty \left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+\int_1^\infty \left(it_2\frac{k^{\gamma_0}}{x^{\gamma_0}-tk^{\gamma_0}\gamma_0}-it_2\frac{k^{\gamma_0}}{x^{\gamma_0}}\right)x^{-2}\,dx+o(k^{-1})\\
&= k^{-1}\int_0^\infty \left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+k^{-1}\int_{1/k}^\infty it_2\frac{t\gamma_0}{x^{\gamma_0}(x^{\gamma_0}-t\gamma_0)}x^{-2}\,dx+o(k^{-1})\\
&= k^{-1}\int_0^\infty \left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+k^{-1}\int_0^\infty it_2\frac{t\gamma_0}{x^{\gamma_0}(x^{\gamma_0}-t\gamma_0)}x^{-2}\,dx+o(k^{-1})
\end{aligned}\tag{A.6}$$

and

$$\begin{aligned}
\delta_3(t_1,t_2) &= E\{\exp(it_1k^{-1/2}l_1(Y_j^*;-1/\gamma_0+tk^{\gamma}))-1\}\{\exp(it_2k^{\gamma_0}l_2(Y_j^*;-1/\gamma_0+tk^{\gamma}))-1\}\\
&= o(k^{-1}).
\end{aligned}\tag{A.7}$$

By (A.5)–(A.7),

$$\begin{aligned}
&E\exp\left\{it_1k^{-1/2}\sum_{j=1}^k l_1(Y_j^*;-1/\gamma_0+tk^{\gamma_0})+it_2k^{\gamma_0}\sum_{j=1}^k l_2(Y_j^*;-1/\gamma_0+tk^{\gamma_0})\right\}\\
&\quad=\{1+\delta_1(t_1)+\delta_2(t_2)+\delta_3(t_1,t_2)\}^k\\
&\quad\to\exp\left\{-\frac{1}{2}t_1^2\gamma_0^2+\int_0^\infty\left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx\right.\\
&\qquad\left.+\int_0^\infty it_2\frac{t\gamma_0}{x^{\gamma_0}(x^{\gamma_0}-t\gamma_0)}x^{-2}\,dx\right\}.
\end{aligned}\tag{A.8}$$

Hence part (i) follows from (A.8).

(ii) To prove the existence and uniqueness of a solution to (A.3), it suffices to show that $\varphi(x)$ is strictly increasing, $\varphi(0+)=-\infty$ and $\varphi((t\gamma_0)^{1/\gamma_0}-)=\infty$. The proof is straightforward and the detail is omitted.

Conditional on $\{(\max_{1\leqslant j\leqslant k}Y_j^*)^{\gamma_0}/\gamma_0 < tk^{\gamma_0}\} = \{\max_{1\leqslant j\leqslant k}Y_j^* < (t\gamma_0)^{1/\gamma_0}k\}$, $Y_1^*,\dots,Y_k^*$ are i.i.d with the truncated density function

$$g_t(y) = \begin{cases} (1-(t\gamma_0)^{-1/\gamma_0}k^{-1})^{-1}y^{-2} & \text{if } 1<y<(t\gamma_0)^{1/\gamma_0}k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z_1,\dots,Z_k$ be iid random variables with the density function $g_t(y)$. Similar to the proof of (A.5), we have

$$\begin{aligned} \bar{\delta}_1(t_1) &:= E\{\exp(it_1k^{-1/2}l_1(Z_j;-1/\gamma_0+tk^{\gamma_0}))\}-1 \\ &= -\tfrac{1}{2}t_1^2\gamma_0^2k^{-1}+o(k^{-1}). \end{aligned} \tag{A.9}$$

It is easy to check that

$$\begin{aligned} \bar{\delta}_2(t_2) &:= E\{\exp(it_2k^{\gamma_0}l_2(Z_j;-1/\gamma_0+tk^{\gamma_0}))\}-1 \\ &= \int_1^{(t\gamma_0)^{1/\gamma_0}k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)\exp\left(-it_2k^{\gamma_0}\frac{1}{1+\gamma_0}\right)-1\right\}\frac{x^{-2}}{1-(t\gamma_0)^{-1/\gamma_0}k^{-1}}\,dx \\ &= \int_{a(t,\gamma_0)k}^{(t\gamma_0)^{1/\gamma_0}k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)\exp\left(-it_2k^{\gamma_0}\frac{1}{1+\gamma_0}\right)-1\right\}\frac{x^{-2}}{1-(t\gamma_0)^{-1/\gamma_0}k^{-1}}\,dx \\ &\quad+\int_1^{a(t,\gamma_0)k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)\exp\left(-it_2k^{\gamma_0}\frac{1}{1+\gamma_0}\right)-1\right\}\frac{x^{-2}}{1-(t\gamma_0)^{-1/\gamma_0}k^{-1}}\,dx \\ &:= I_1+I_2, \end{aligned} \tag{A.10}$$

where

$$I_1 = k^{-1}\int_{a(t,\gamma_0)}^{(t\gamma_0)^{1/\gamma_0}}\left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1\right\}x^{-2}\,dx+o(k^{-1}), \tag{A.11}$$

and

$$\begin{aligned} I_2 &= \int_1^{a(t,\gamma_0)k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)-1-it_2k^{\gamma_0}(1+\gamma_0)^{-1}\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)\right\}x^{-2}\,dx+o(k^{-1}) \\ &= \int_1^{a(t,\gamma_0)k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)-1-it_2k^{\gamma_0}(1+\gamma_0)^{-1}\right\}x^{-2}\,dx+o(k^{-1}) \\ &= \int_1^{a(t,\gamma_0)k}\left\{\exp\left(it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right)-1-it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}\right\}x^{-2}\,dx \\ &\quad+\int_1^{a(t,\gamma_0)k}\left\{it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}-it_2k^{\gamma_0}(1+\gamma_0)^{-1}\right\}x^{-2}\,dx+o(k^{-1}) \\ &= k^{-1}\int_0^{a(t,\gamma_0)}\left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+\int_1^{a(t,\gamma_0)k}\left\{it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}-it_2k^{\gamma_0}\frac{1}{x^{\gamma_0}}\right\}x^{-2}\,dx \\ &\quad+\int_1^{a(t,\gamma_0)k}\left\{it_2k^{\gamma_0}\frac{1}{x^{\gamma_0}}-it_2k^{\gamma_0}(1+\gamma_0)^{-1}\right\}x^{-2}\,dx+o(k^{-1}) \\ &:= k^{-1}\int_0^{a(t,\gamma_0)}\left\{\exp\left(it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right)-1-it_2\frac{1}{x^{\gamma_0}-t\gamma_0}\right\}x^{-2}\,dx+I_3+I_4+o(k^{-1}). \end{aligned}$$

Notice that

$$\begin{aligned} I_3 &= \int_1^{a(t,\gamma_0)k}\left\{it_2k^{\gamma_0}\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0k^{\gamma_0}}-it_2k^{\gamma_0}\frac{1}{x^{\gamma_0}}\right\}x^{-2}\,dx \\ &= k^{-1}\int_{1/k}^{a(t,\gamma_0)}\left\{it_2\frac{1-t\gamma_0k^{\gamma_0}}{x^{\gamma_0}-t\gamma_0}-it_2\frac{1}{x^{\gamma_0}}\right\}x^{-2}\,dx \\ &= k^{-1}it_2\int_{1/k}^{a(t,\gamma_0)}\left\{\frac{1}{x^{\gamma_0}-t\gamma_0}-\frac{1}{x^{\gamma_0}}\right\}x^{-2}\,dx+o(k^{-1}) \\ &= k^{-1}it_2\int_0^{a(t,\gamma_0)}\frac{t\gamma_0}{(x^{\gamma_0}-t\gamma_0)x^{\gamma_0}}x^{-2}\,dx+o(k^{-1}) \end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \int_1^{a(t,\gamma_0)k} \left\{ it_2 k^{\gamma_0} \frac{1}{x^{\gamma_0}} - it_2 k^{\gamma_0}(1+\gamma_0)^{-1} \right\} x^{-2}\,dx \\
&= k^{-1} \int_{1/k}^{a(t,\gamma_0)} it_2 \frac{1}{x^{\gamma_0}} x^{-2}\,dx + o(k^{-1}) \\
&= k^{-1} \frac{a(t,\gamma_0)^{-\gamma_0+1}}{\gamma_0+1} + o(k^{-1}).
\end{aligned}
$$

From the definition of $a(t,\gamma_0)$, we get that $I_3 + I_4 = o(k^{-1})$. Thus

$$
I_2 = k^{-1} \int_0^{a(t,\gamma_0)} \left\{ \exp\left(it_2 \frac{1}{x^{\gamma_0} - t\gamma_0} \right) - 1 - it_2 \frac{1}{x^{\gamma_0} - t\gamma_0} \right\} x^{-2}\,dx + o(k^{-1}). \tag{A.12}
$$

Furthermore, similar to the proof of part (i), we have that

$$
\begin{aligned}
\bar{\delta}_3(t_1,t_2) &= E\{\exp(it_1 k^{-1/2} l_1(Z_j; -1/\gamma_0 + tk^{\gamma})) - 1\}\{\exp(it_2 k^{\gamma_0} l_2(Z_j; -1/\gamma_0 + tk^{\gamma})) - 1\} \\
&= o(k^{-1}).
\end{aligned} \tag{A.13}
$$

By (A.9)–(A.13),

$$
\begin{aligned}
& E\exp\left\{ it_1 k^{-1/2} \sum_{j=1}^k l_1(Z_j; -1/\gamma_0 + tk^{\gamma_0}) + it_2 k^{\gamma_0} \sum_{j=1}^k l_2(Z_j; -1/\gamma_0 + tk^{\gamma_0}) \right\} \\
&\quad = \{1 + \bar{\delta}_1(t_1) + \bar{\delta}_2(t_2) + \bar{\delta}_3(t_1,t_2)\}^k \\
&\quad \to \exp\left\{ -\frac{1}{2} t_1^2 \gamma_0^2 + \int_0^{a(t,\gamma_0)} \left\{ \exp\left(it_2 \frac{1}{x^{\gamma_0} - t\gamma_0} \right) - 1 - it_2 \frac{1}{x^{\gamma_0} - t\gamma_0} \right\} x^{-2}\,dx \right. \\
&\qquad \left. + \int_{a(t,\gamma_0)}^{(t\gamma_0)^{1/\gamma_0}} \left\{ \exp\left(it_2 \frac{1}{x^{\gamma_0} - t\gamma_0} \right) - 1 \right\} x^{-2}\,dx \right\}.
\end{aligned} \tag{A.14}
$$

Hence part (ii) follows from (A.14).

Part (iii) and Part (iv) can be shown in a similar way to the proofs of (i) and (ii), respectively. Hence, we complete the proof the lemma. □

Lemma 3. *Assume $\gamma_0 = -\frac{1}{2}$.*

(i) *For any $t \geqslant 0$ we have*

$$
\left(\frac{1}{\sqrt{k}} \sum_{j=1}^k l_1(Y_j^*; -1/\gamma_0 + t(k\ln k)^{\gamma_0}), \frac{1}{\sqrt{k\ln k}} \sum_{j=1}^k l_2(Y_j^*; -1/\gamma_0 + t(k\ln k)^{\gamma_0}) \right) \xrightarrow{d} (N_1, N_2) \tag{A.15}
$$

as $k \to \infty$, where N_1 and N_2 are independent normal random variables with $N_1 \sim \mathrm{N}(0,\gamma_0^2)$ and $N_2 \sim \mathrm{N}(\gamma_0 t, 1)$.

(ii) *For each $t < 0$, $P((\max_{1\leqslant j\leqslant k} Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}) \to 1$ as $k \to \infty$, and conditional on $\{(\max_{1\leqslant j\leqslant k} Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}\}$, (A.15) holds with N_1 and N_2 given in part (i).*

(iii) *For any $t \geqslant 0$ we have*

$$
\left(\frac{1}{\sqrt{k}} \sum_{j=1}^k l_1(Y_j^*; -1/\gamma_0), \frac{1}{\sqrt{k\ln k}} \sum_{j=1}^k l_2(Y_j^*; -1/\gamma_0 + t(k\ln k)^{\gamma_0}) \right) \xrightarrow{d} (N_1, N_2) \tag{A.16}
$$

as $k \to \infty$, where N_1 and N_2 are given in (i).

(iv) *For each $t < 0$, $P((\max_{1\leqslant j\leqslant k} Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}) \to 1$ as $k \to \infty$, and conditional on $\{(\max_{1\leqslant j\leqslant k} Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}\}$, (A.16) holds with N_1 and N_2 given in part (i).*

Proof. Set

$$
V_{1,j} := \frac{l_1\left(Y_j^*; -1/\gamma_0 + \dfrac{t}{\sqrt{k\ln k}} \right) I\left(Y_j^* \leqslant k(\ln k)^{1/2} \right)}{\sqrt{k}}
$$

and

$$
V_{2,j} := \frac{l_2\left(Y_1^*; -1/\gamma_0 + \dfrac{t}{\sqrt{k\ln k}} \right) I\left(Y_j^* \leqslant k(\ln k)^{1/2} \right)}{\sqrt{k\ln k}},
$$

which are uniformly bounded by a sequence of constants tending to zero. Moreover, by noting $\gamma_0 = -\frac{1}{2}$, one can verify that $E(V_{1,1}) = o(1/k)$, $E(V_{2,1}) = \gamma_0 t/k + o(1/k)$, $E(V_{1,1}V_{2,1}) = o(1/k)$, $EV_{1,1}^2 = 1/4k + o(1/k)$, and $EV_{2,1}^2 = 1/k + o(1/k)$. By using Taylor's expansion we can show that

$$E\exp\left(it_1\sum_{j=1}^k V_{1,j} + it_2\sum_{j=1}^k V_{2,j}\right) \to \exp\left(-\frac{1}{8}t_1^2 + it_2\gamma_0 t - \frac{1}{2}t_2^2\right).$$

Therefore, $(\sum_{j=1}^k V_{1,j}, \sum_{j=1}^k V_{2,j}) \xrightarrow{d} (N_1, N_2)$, and part (i) is proved since $P(\cup_{j=1}^k\{Y_j^* > k(\ln k)^{1/2}\}) \leqslant kP(Y_1^* > k(\ln k)^{1/2}) = (\ln k)^{-1/2} \to 0$ as $k \to \infty$.

For each given $t < 0$, we have

$$P\left(\left(\max_{1\leqslant j\leqslant k}Y_j^*\right)^{\gamma_0}/\gamma_0 \geqslant t(k\ln k)^{\gamma_0}\right) = P\left(\max_{1\leqslant j\leqslant k}Y_j^* \geqslant (t\gamma_0)^{1/\gamma_0}k\ln k\right)$$
$$\leqslant kP(Y_1^* \geqslant (t\gamma_0)^{1/\gamma_0}k\ln k) = O((\ln k)^{-1}) \to 0,$$

yielding $P((\max_{1\leqslant j\leqslant k}Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}) \to 1$.

The rest of the proof for part (ii) is the same as that of part (i) by replacing expectations and probabilities by conditional expectations and conditional probabilities given on the set $\{(\max_{1\leqslant j\leqslant k}Y_j^*)^{\gamma_0}/\gamma_0 < t(k\ln k)^{\gamma_0}\}$.

Parts (iii) and (iv) can be proved in a way similar to the proofs of (i) and (ii). Hence, we complete the proof of Lemma 3.

Proof of Proposition 1. Put

$$\bar{f}_n(t) = \frac{1}{k}\sum_{i=1}^k \log\left\{1 - \frac{(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0} - 1}{t\gamma_0}\right\} + 1,$$

$$\bar{g}_n(t) = \frac{1}{k}\sum_{i=1}^k \left\{1 - \frac{(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0} - 1}{t\gamma_0}\right\}^{-1},$$

$$\bar{l}_n(t) = \frac{1}{k}\sum_{i=1}^k \left\{1 - \frac{(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0} - 1}{t\gamma_0}\right\}^{-2},$$

$$\bar{M}_n = \frac{(Y_{n,n}/Y_{n,n-k})^{\gamma_0} - 1}{\gamma_0}$$

and

$$\bar{h}_n(t) = \bar{g}_n(t) - \frac{1}{\bar{f}_n(t)}.$$

By (2.1), (2.5) and Lemma 1, there exist d_1 and d_2 such that

$$\frac{(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0} - 1}{(Y_{n,n}/Y_{n,n-k})^{\gamma_0} - 1}(1 + d_1A(n/k)) \leqslant \frac{X_{n,n-i+1} - X_{n,n-k}}{X_{n,n} - X_{n,n-k}} \leqslant \frac{(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0} - 1}{(Y_{n,n}/Y_{n,n-k})^{\gamma_0} - 1}(1 + d_2A(n/k)) \tag{A.17}$$

with probability tending to one.

Note that (2.8) implies that $\delta_n/|A(n/k)| \to \infty$ and $k^{-\gamma_0}\delta_n \to 0$ as $n \to \infty$, and it is easy to verify that

$$k^{-\gamma_0}(\bar{M}_n + 1/\gamma_0) \xrightarrow{d} -Q, \quad k^{-\gamma_0}(\bar{M}_n(1+\delta_n) + 1/\gamma_0) \xrightarrow{d} -Q, \tag{A.18}$$

where Q is a positive random variable with distribution $1 - \exp(-(-\gamma_0 x)^{-1/\gamma_0})$, $x > 0$. Further, we can show that

$$(f_n(tM_n(1+\delta_n)),\ g_n(tM_n(1+\delta_n)),\ M_n) = (\bar{f}_n(t\bar{M}_n(1+\delta_n)),\ \bar{g}_n(t\bar{M}_n(1+\delta_n)),\ \bar{M}_n)(1 + O_p(\delta_n)) \tag{A.19}$$

uniformly for $t \geqslant 1$. Hence, by (A.19), (A.18) and the weak law of large numbers we have

$$\begin{aligned} h_n(-1/\gamma_0 + \varepsilon) &= \bar{g}_n(-1/\gamma_0 + \varepsilon) - \bar{f}_n^{-1}(-1/\gamma_0 + \varepsilon) + o_p(1) \\ &\xrightarrow{p} E\left\{\frac{1 - \varepsilon\gamma_0}{(Y_1^*)^{\gamma_0} - \varepsilon\gamma_0}\right\} - \frac{1}{E\log\left\{\frac{(Y_1^*)^{\gamma_0} - \varepsilon\gamma_0}{1 - \varepsilon\gamma_0}\right\} + 1} \\ &:= J_1 - J_2. \end{aligned} \tag{A.20}$$

It is easy to check that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\left\{J_1-\frac{1}{1+\gamma_0}\right\}&=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_1^\infty\left\{\frac{1-\varepsilon\gamma_0}{y^{\gamma_0}-\varepsilon\gamma_0}-\frac{1}{y^{\gamma_0}}\right\}y^{-2}\,dy\\&=\lim_{\varepsilon\downarrow 0}\int_1^\infty\frac{\gamma_0(1-y^{\gamma_0})}{y^{\gamma_0}(y^{\gamma_0}-\varepsilon\gamma_0)}y^{-2}\,dy\\&=-\infty\end{aligned}\tag{A.21}$$

and

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\{J_2^{-1}-(1+\gamma_0)\}&=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\left\{\int_1^\infty\log\left(\frac{y^{\gamma_0}-\varepsilon\gamma_0}{1-\varepsilon\gamma_0}\right)d(-y^{-1})-\gamma_0\right\}\\&=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\gamma_0\left\{\int_1^\infty\frac{1-\varepsilon\gamma_0}{y^{\gamma_0}-\varepsilon\gamma_0}y^{\gamma_0-2}\,dy-1\right\}\\&=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\gamma_0\int_1^\infty\left\{\frac{1-\varepsilon\gamma_0}{y^{\gamma_0}-\varepsilon\gamma_0}-\frac{1}{y^{\gamma_0}}\right\}y^{\gamma_0-2}\,dy\\&=\lim_{\varepsilon\downarrow 0}\int_1^\infty\frac{\gamma_0^2(1-y^{\gamma_0})}{y^{\gamma_0}(y^{\gamma_0}-\varepsilon\gamma_0)}y^{\gamma_0-2}\,dy\\&=\gamma_0^2\int_1^\infty(1-y^{\gamma_0})y^{-2-\gamma_0}\,dy\\&=-\frac{\gamma_0^3}{1+\gamma_0},\end{aligned}$$

which implies

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\left\{J_2-\frac{1}{1+\gamma_0}\right\}=\frac{\gamma_0^3}{(1+\gamma_0)^3}.\tag{A.22}$$

Thus, (2.10) follows from Eqs. (A.20) to (A.22).

Next we prove that uniformly in $t\in[\bar{M}_n(1+k^{-1}),-1/\gamma_0]$,

$$\bar{h}_n'(t)/d_n\xrightarrow{p}-\infty\tag{A.23}$$

for $d_n=k^{-3(1+2\gamma_0)/4}(\ln k)^{3/4}$. Note that $d_n\to\infty$ as $n\to\infty$.

Note that

$$\begin{aligned}\bar{h}_n'(t)&=t^{-1}\{-\bar{l}_n(t)+\bar{g}_n(t)+\bar{f}_n^{-2}(t)\bar{g}_n(t)-\bar{f}_n^{-2}(t)\}\\&\leqslant t^{-1}\{-\bar{l}_n(t)+\bar{g}_n(t)+\bar{f}_n^{-2}(t)\bar{g}_n(t)\}\\&\leqslant t^{-1}\{-\bar{l}_n(t)+\bar{g}_n(t)+\bar{f}_n^{-2}(\bar{M}_n(1+k^{-1}))\bar{g}_n(t)\}\\&\leqslant t^{-1}\{-\bar{l}_n(t)+(1+c)\bar{g}_n(t)\}\end{aligned}$$

holds with a probability tending to one, where $c=1/(1+\gamma_0+\varepsilon)^2>0$ for some sufficiently small $\varepsilon>0$.

Denote $Z_{n,i}(t)=1-((Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0}-1)/t\gamma_0$ for $1\leqslant i\leqslant k$. For all $t\in[\bar{M}_n(1+k^{-1}),-1/\gamma_0]$, we have uniformly

$$Z_{n,i}(t)\leqslant Z_{n,i}(-1/\gamma_0)=(Y_{n,n-i+1}/Y_{n,n-k})^{\gamma_0}.$$

Therefore, by choosing $0<b<(1+c)^{1/\gamma_0}$, for all $i\leqslant\lfloor bk\rfloor$, where $\lfloor bk\rfloor$ denotes the integer part of bk, we have

$$Z_{n,i}(t)\leqslant(Y_{n,n-\lfloor bk\rfloor+1}/Y_{n,n-k})^{\gamma_0}\xrightarrow{p}b^{-\gamma_0}<\frac{1}{1+c}.$$

Denote

$$I_1(t)=\frac{1}{k}\sum_{i=1}^{\lfloor bk\rfloor}\frac{1}{(Z_{n,i}(t))^2},\quad I_2(t)=\frac{1}{k}\sum_{i=\lfloor bk\rfloor+1}^{k}\frac{1}{Z_{n,i}(t)}.$$

For all $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$, we have uniformly

$$\begin{aligned} I_1(t) &\geqslant I_1(-1/\gamma_0) \\ &= \frac{1}{k}\sum_{i=1}^{\lfloor bk \rfloor} \frac{1}{(Z_{n,i}(-1/\gamma_0))^2} \\ &= \frac{1}{k}\sum_{i=1}^{\lfloor bk \rfloor} (Y_{n,n-i+1}/Y_{n,n-k})^{-2\gamma_0}. \end{aligned}$$

Note that $(1/k)\sum_{i=\lfloor bk \rfloor+1}^{k}(Y_{n,n-i+1}/Y_{n,n-k})^{-2\gamma_0}$ is bounded by a positive number. It follows from (A.1) that if $\gamma_0 \in (-1, -\frac{1}{2})$

$$\frac{1}{k^{-2\gamma_0}}\sum_{i=1}^{k}(Y_{n,n-i+1}/Y_{n,n-k})^{-2\gamma_0} \stackrel{d}{=} \frac{1}{k^{-2\gamma_0}}\sum_{i=1}^{k}(Y_i^*)^{-2\gamma_0} \stackrel{d}{\to} Q_1,$$

where $Q_1 > 0$ is a random variable with $1/(-2\gamma_0)$-stable distribution. Similarly, for $\gamma_0 = -\frac{1}{2}$, $(1/k)(\sum_{i=1}^{k}(Y_{n,n-i+1}/Y_{n,n-k}) - k\ln k)$ converges in distribution. Therefore, for any $\gamma_0 \in (-1, -\frac{1}{2}]$, we have $(1/kd_n)s\sum_{i=1}^{k}(Y_{n,n-i+1}/Y_{n,n-k})^{-2\gamma_0}$ converges to infinity in probability, where d_n is given in (A.23). Thus, we conclude that uniformly in $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$,

$$I_1(t)/d_n \stackrel{p}{\to} +\infty. \tag{A.24}$$

In a similar manner, we have uniformly in $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$

$$\begin{aligned} I_2(t) &\leqslant I_2(\bar{M}_n(1+k^{-1})) \\ &= \frac{1}{k}\sum_{i=\lfloor bk \rfloor+1}^{k} \frac{1}{Z_{n,i}(\bar{M}_n(1+k^{-1}))} \\ &\leqslant \frac{k-\lfloor bk \rfloor}{k}\frac{1}{Z_{n,\lfloor bk \rfloor+1}(\bar{M}_n(1+k^{-1}))} \\ &\stackrel{p}{\to} (1-b)b^{-\gamma_0}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} -\bar{l}_n(t) + (1+c)\bar{g}_n(t) &= \frac{1}{k}\sum_{i=1}^{k} \frac{1+c}{Z_{n,i}(t)} - \frac{1}{Z_{n,i}(t)^2} \\ &\leqslant \frac{1}{k}\sum_{i=1}^{\lfloor bk \rfloor} \frac{(1+c)(Z_{n,i}(t)) - 1}{Z_{n,i}(t)^2} + \frac{1}{k}\sum_{i=\lfloor bk \rfloor+1}^{k} \frac{1+c}{Z_{n,i}(t)} \\ &= -\frac{1}{k}\sum_{i=1}^{\lfloor bk \rfloor}(1-(1+c)(Z_{n,i}(t))) \cdot \frac{1}{Z_{n,i}(t)^2} + (1+c)I_2(t). \end{aligned}$$

Notice that for any $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$, the two series $1-(1+c)(Z_{n,i}(t))$ and $1/Z_{n,i}(t)^2$ are both positive and decreasing in i when $1 \leqslant i \leqslant \lfloor bk \rfloor$. By using the Chebyshev's inequality with ordering series (see Jeffrey, 1995, p. 28), we get that

$$\begin{aligned} \frac{1}{k}\sum_{i=1}^{\lfloor bk \rfloor}(1-(1+c)(Z_{n,i}(t))) \cdot \frac{1}{(Z_{n,i}(t))^2} &\geqslant \frac{\lfloor bk \rfloor}{k}\frac{1}{\lfloor bk \rfloor}\sum_{i=1}^{\lfloor bk \rfloor}(1-(1+c)(Z_{n,i}(t))) \cdot \frac{1}{\lfloor bk \rfloor}\sum_{i=1}^{\lfloor bk \rfloor}\frac{1}{(Z_{n,i}(t))^2} \\ &\geqslant (1-(1+c)(Z_{n,\lfloor bk \rfloor}(-1/\gamma_0))) \cdot I_1(t). \end{aligned}$$

Since $(1-(1+c)(Z_{n,\lfloor bk \rfloor}(-1/\gamma_0))) \stackrel{p}{\to} (1-(1+c)b^{-\gamma_0}) > 0$, we conclude that uniformly in $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$

$$\frac{1}{kd_n}\sum_{i=1}^{\lfloor bk \rfloor}(1-(1+c)(Z_{n,i}(t))) \cdot \frac{1}{(Z_{n,i}(t))^2} \stackrel{p}{\to} +\infty,$$

where d_n is given in (A.23). Since $I_2(t)$ is bounded in probability by a finite number, we have proved that uniformly in $t \in [\bar{M}_n(1+k^{-1}), -1/\gamma_0]$

$$\{-\bar{l}_n(t) + (1+c)\bar{g}_n(t)\}/d_n \stackrel{p}{\to} -\infty. \tag{A.25}$$

This proves (A.23).

Since

$$\bar{l}_n(-1/\gamma_0+\varepsilon)\xrightarrow{p} E\left\{\frac{1-\varepsilon\gamma_0}{(Y_1^*)^{\gamma_0}-\varepsilon\gamma_0}\right\}^2,$$

$$\lim_{\varepsilon\downarrow 0} E\left\{\frac{1-\varepsilon\gamma_0}{(Y_1^*)^{\gamma_0}-\varepsilon\gamma_0}\right\}^2=\infty,\quad \bar{g}_n(-1/\gamma_0)\xrightarrow{p}(1+\gamma_0)^{-1},$$

we have, uniformly in $t\in[-1/\gamma_0,-1/\gamma_0+\varepsilon]$,

$$\begin{aligned}\bar{h}_n'(t)&\leqslant t^{-1}\{-\bar{l}_n(-1/\gamma_0+\varepsilon)+\bar{g}_n(-1/\gamma_0)+\bar{f}_n^{-2}(-1/\gamma_0)\bar{g}_n(-1/\gamma_0)-\bar{f}_n^{-2}(-1/\gamma_0+\varepsilon)\}\\ &<0\end{aligned}\tag{A.26}$$

with a probability tending to one when $\varepsilon>0$ is small enough. Therefore, (2.11) follows from (A.23), (A.26), (2.8) and (A.19).

Next we prove (2.9). By the mean-value theorem,

$$\bar{h}_n(\bar{M}_n(1+\delta_n))-\bar{h}_n(-1/\gamma_0)=\bar{h}_n'(\xi)\{\bar{M}_n(1+\delta_n)+1/\gamma_0\}\tag{A.27}$$

for some ξ between $\bar{M}_n(1+\delta_n)$ and $-1/\gamma_0$. It follows from Lemmas 2 and 3 that

$$\bar{h}_n(-1/\gamma_0)=O_p(k^{-1/2})\quad\text{for }\gamma_0\in(-1,-1/2)$$

and

$$\bar{h}_n(-1/\gamma_0)=O_p\left(\left(\frac{k}{\ln k}\right)^{-1/2}\right)\quad\text{for }\gamma_0=-1/2.$$

Since

$$k^{-1/2}/(d_nk^{\gamma_0})\to 0\quad\text{for }\gamma_0\in(-1,-1/2)$$

and

$$(k/\ln k)^{-1/2}/(d_nk^{\gamma_0})\to 0\quad\text{for }\gamma_0=-1/2,$$

where d_n is given in (A.23), we get

$$\frac{\bar{h}_n(-1/\gamma_0)}{d_nk^{\gamma_0}}\xrightarrow{p}0.\tag{A.28}$$

Moreover, we have from Eqs. (A.18) and (A.23) that

$$\frac{\bar{h}_n'(\xi)\{\bar{M}_n(1+\delta_n)+1/\gamma_0\}}{d_nk^{\gamma_0}}=\frac{\bar{h}_n'(\xi)}{d_n}\,\frac{\bar{M}_n(1+\delta_n)+1/\gamma_0}{k^{\gamma_0}}\xrightarrow{p}\infty,$$

which together with (A.27) and (A.28) implies

$$\bar{h}_n(\bar{M}_n(1+\delta_n))/(d_nk^{\gamma_0})\to+\infty\tag{A.29}$$

with probability tending to one.

It follows from (A.19) that

$$h_n(M_n(1+\delta_n))=\bar{h}_n(\bar{M}_n(1+\delta_n))(1+O_p(\delta_n))+O_p(\delta_n),$$

which coupled with (A.29) implies

$$h_n(M_n(1+\delta_n))/(d_nk^{\gamma_0})\xrightarrow{p}+\infty$$

and hence (2.9). Therefore we complete the proof of Proposition 1. □

For proving Theorems 1 and 2, we introduce some notations. For any two sets A and B denote the symmetric difference of A and B by $A\Delta B$, i.e. $A\Delta B=(A\cup B)\backslash(A\cap B)$. The following facts about the symmetric differences will be used in the proofs:

D1. $P(A_n\Delta B_n)\to 0$ implies $P(A_n)-P(B_n)\to 0$;
D2. If $P(A_n\Delta B_n)\to 0$ as $n\to\infty$, then $P(A_n\cap C_n)-P(A_n)P(C_n)\to 0$ if and only if $P(B_n\cap C_n)-P(B_n)P(C_n)\to 0$.

From Proposition 1 we conclude that

$$P(\{\hat{\tau}/a(U^{-}(X_{n,n-k}))>t\}\Delta\{h_n(t)>0\})=o(1), \tag{A.30}$$

where o(1) is uniform for all $-(1+\delta_n)/\gamma_0 \leqslant t \leqslant -1/\gamma_0+\varepsilon$, and

$$P(\{\hat{\tau}/a(U^{-}(X_{n,n-k}))\leqslant t\}\Delta\{h_n(t)\leqslant 0, M_n(1+\delta_n)<t\})=o(1),$$

where o(1) is uniform for all $t\leqslant -(1+\delta_n)/\gamma_0$. Therefore, for $-1<\gamma_0<-\frac{1}{2}$ we have

$$\begin{aligned} P(\{k^{-\gamma_0}(\hat{\tau}/a(U^{-}(X_{n,n-k}))+1/\gamma_0)>t\}\Delta\{h_n(-1/\gamma_0+tk^{\gamma_0})>0\}) &= P(\{\hat{\tau}/a(U^{-}(X_{n,n-k}))>-1/\gamma_0+tk^{\gamma_0}\}\Delta\{h_n(-1/\gamma_0+tk^{\gamma_0})>0\}) \\ &= o(1) \end{aligned} \tag{A.31}$$

for $t>0$, and

$$\begin{aligned} &P(\{k^{-\gamma_0}(\hat{\tau}/a(U^{-}(X_{n,n-k}))+1/\gamma_0)\leqslant t\}\Delta\{h_n(-1/\gamma_0+tk^{\gamma_0})\leqslant 0, M_n(1+\delta_n)<-1/\gamma_0+tk^{\gamma_0}\}) \\ &\quad = P(\{\hat{\tau}/a(U^{-}(X_{n,n-k}))\leqslant -1/\gamma_0+tk^{\gamma_0}\}\Delta\{h_n(-1/\gamma_0+tk^{\gamma_0})\leqslant 0, M_n(1+\delta_n)<-1/\gamma_0+tk^{\gamma_0}\}) \\ &\quad = o(1) \end{aligned} \tag{A.32}$$

for $t<0$. Similarly, for $\gamma_0=-\frac{1}{2}$ we have

$$P(\{(k\ln k)^{-\gamma_0}(\hat{\tau}/a(U^{-}(X_{n,n-k}))+1/\gamma_0)>t\}\Delta\{h_n(-1/\gamma_0+t(k\ln k)^{\gamma_0})>0\})=o(1) \tag{A.33}$$

for $t\geqslant 0$, and

$$P\left(\left\{(k\ln k)^{-\gamma_0}\left(\frac{\hat{\tau}}{a(U^{-}(X_{n,n-k}))}+1/\gamma_0\right)\leqslant t\right\}\Delta\{h_n(-1/\gamma_0+t(k\ln k)^{\gamma_0})\leqslant 0, M_n(1+\delta_n)<-1/\gamma_0+t(\ln k)^{\gamma_0}\}\right)=o(1) \tag{A.34}$$

for $t<0$.

Proof of Theorem 1. By Lemma 2, (A.1) and (A.19), we have for any $t>0$

$$\begin{aligned} \lim_{n\to\infty} P(h_n(-1/\gamma_0+tk^{\gamma_0})>0) &= \lim_{n\to\infty} P(k^{1+\gamma_0}h_n(-1/\gamma_0+tk^{\gamma_0})>0) \\ &= \lim_{n\to\infty} P(k^{1+\gamma_0}(\bar{h}_n(-1/\gamma_0+tk^{\gamma_0})>0) \\ &= \lim_{n\to\infty} P\left(k^{1+\gamma_0}\left(\bar{g}_n(-1/\gamma_0+tk^{\gamma_0})-\frac{1}{1+\gamma_0}\right)>0\right) \\ &= P(W_t>0). \end{aligned} \tag{A.35}$$

Since

$$\lim_{n\to\infty} P(k^{-\gamma_0}(\bar{M}_n+1/\gamma_0)\leqslant x)=\exp(-(\gamma_0 x)^{-1/\gamma_0}) \quad \text{for } x<0,$$

it follows from (A.19) that $\bar{M}_n=M_n(1+\delta_n)+O_p(\delta_n)$, which implies $E|I(k^{-\gamma_0}(\bar{M}_n+1/\gamma_0)<t)-I(k^{-\gamma_0}(M_n(1+\delta_n)+1/\gamma_0)<t)|\to 0$ for any fixed $t<0$. Therefore, for any $t<0$, Lemma 2 implies that

$$\begin{aligned} &\lim_{n\to\infty} P(h_n(-1/\gamma_0+tk^{\gamma_0})\leqslant 0, \quad M_n(1+\delta_n)<-1/\gamma_0+tk^{\gamma_0}) \\ &\quad = \lim_{n\to\infty} P(k^{1+\gamma_0}\bar{h}_n(-1/\gamma_0+tk^{\gamma_0})\leqslant 0, \quad \bar{M}_n<-1/\gamma_0+tk^{\gamma_0}) \\ &\quad = \lim_{n\to\infty} P(k^{1+\gamma_0}\bar{h}_n(-1/\gamma_0+tk^{\gamma_0})\leqslant 0|\bar{M}_n<-1/\gamma_0+tk^{\gamma_0})P(\bar{M}_n<-1/\gamma_0+tk^{\gamma_0}) \\ &\quad = \lim_{n\to\infty} P\left(k^{1+\gamma_0}\left(\bar{g}_n(-1/\gamma_0+tk^{\gamma_0})-\frac{1}{1+\gamma_0}\right)\leqslant 0|\bar{M}_n<-1/\gamma_0+tk^{\gamma_0}\right)P(\bar{M}_n<-1/\gamma_0+tk^{\gamma_0}) \\ &\quad = P(W_t\leqslant 0)\exp\{-(t\gamma_0)^{-1/\gamma_0}\}. \end{aligned} \tag{A.36}$$

Hence, it follows from (A.31), (A.32), (A.35) and (A.36) that

$$P(k^{-\gamma_0}\{\hat{\tau}/a(U^{-}(X_{n,n-k}))+\gamma_0^{-1}\}\leqslant x)\to V(x), \tag{A.37}$$

which implies that

$$\hat{\tau}/a(U^{-}(X_{n,n-k}))+\gamma_0^{-1}=o_p(k^{-1/2}). \tag{A.38}$$

Note that $g_n(t)$ is monotone in t and $f_n'(t)=(g_n(t)-1)/t$. By the mean-value theorem there exists a variable ξ_n between $\hat{\tau}/a(U^{-}(X_{n,n-k}))$ and $-\gamma_0^{-1}$ such that

$$\begin{aligned}|f_n(\hat{\tau}/a(U^{-}(X_{n,n-k}))-f_n(-\gamma_0^{-1})| &= |f_n'(\xi_n)||\hat{\tau}/a(U^{-}(X_{n,n-k}))+\gamma_0^{-1})| \\ &\leqslant (1+\max(g_n(\hat{\tau}/a(U^{-}(X_{n,n-k})),g_n(-\gamma_0^{-1})))o_p(k^{-1/2}) \\ &= \left(1+\max\left(\frac{1}{1+\hat{\gamma}},g_n(-\gamma_0^{-1})\right)\right)o_p(k^{-1/2}).\end{aligned}$$

Here we used (2.3) and (A.38). From Zhou (2009), $\hat{\gamma}\xrightarrow{p}\gamma_0$. By using (A.19) and Lemma 2(i) (with $t=0$) we have $g_n(-\gamma_0^{-1})=\bar{g}_n(-\gamma_0^{-1})+O_p(\delta_n)\xrightarrow{p}1/(1+\gamma_0)$. Therefore, from (2.2) and Lemma 2 we have

$$\begin{aligned}\sqrt{k}\{\hat{\gamma}-\gamma_0\} &= \sqrt{k}\{f_n(-1/\gamma_0)-1-\gamma_0\}+o_p(1) \\ &= \sqrt{k}\{\bar{f}_n(-1/\gamma_0)-1-\gamma_0\}+o_p(1) \\ &\xrightarrow{d} W.\end{aligned}\tag{A.39}$$

We still need to prove the asymptotic independence. It suffices to show that

$$P(\sqrt{k}\{\hat{\gamma}-\gamma_0\}\leqslant x, k^{-\gamma_0}\{\hat{\tau}/a(U^{-}(X_{n,n-k}))+\gamma_0^{-1}\}\leqslant t)\to P(W\leqslant x)P(V\leqslant t)$$

for $x\in R$ and $t\in R$. Note that the above equation is equivalent to

$$P(\sqrt{k}\{\hat{\gamma}-\gamma_0\}\leqslant x, k^{-\gamma_0}\{\hat{\tau}/a(U^{-}(X_{n,n-k}))+\gamma_0^{-1}\}> t)\to P(W\leqslant x)P(V> t).$$

Here we focus on the case $t>0$. By using (A.31) it suffices to show the asymptotic independence of $\sqrt{k}\{\hat{\gamma}-\gamma_0\}$ and $k^{1+\gamma_0}h_n(-1/\gamma_0+tk^{\gamma_0})$. From Lemma 2(i) we know that $\bar{f}_n(-1/\gamma_0+tk^{\gamma_0})-1-\gamma_0$ is negligible compared with $\bar{g}_n(-1/\gamma_0+tk^{\gamma_0})-1/(1+\gamma_0)$. Hence, from (A.39) and (A.19) we have that

$$\begin{aligned}(\sqrt{k}\{\hat{\gamma}-\gamma_0\},k^{1+\gamma_0}h_n(-1/\gamma_0+tk^{\gamma_0})) &= (\sqrt{k}(\bar{f}_n(-1/\gamma_0)-1-\gamma_0),k^{1+\gamma_0}\bar{h}_n(-1/\gamma_0+tk^{\gamma_0}))+o_p(1) \\ &= (\sqrt{k}(\bar{f}_n(-1/\gamma_0)-1-\gamma_0),k^{1+\gamma_0}(\bar{g}_n(-1/\gamma_0+tk^{\gamma_0})-1/(1+\gamma_0)))+o_p(1).\end{aligned}$$

So the independence for the case $t\geqslant 0$ follows from part (iii) of Lemma 2. Similarly, we can show the independence for the case $t<0$ by using parts (ii) and (iv) of Lemma 2. We complete the proof of Theorem 1. □

Proof of Theorem 2. Following the same lines of the proof of Theorem 1, this theorem can be proved by using Lemma 3, Eqs. (A.33) and (A.34). □

Proof of Theorem 3. From Lemma 4.1 of Ferreira et al. (2003), we have

$$\lim_{t\to\infty}\frac{\dfrac{U(\infty)-U(t)}{a(t)}+\gamma_0^{-1}}{A(t)}=\frac{-1}{\gamma_0(\gamma_0+\rho)}.\tag{A.40}$$

Write

$$\begin{aligned}\frac{a(U^{-}(X_{n,n-k}))}{a(n/k)}-1 &= \left\{\frac{a(U^{-}(X_{n,n-k}))}{U(\infty)-X_{n,n-k}}+\gamma_0\right\}\frac{U(\infty)-X_{n,n-k}}{a(n/k)}-\gamma_0\left\{\frac{U(\infty)-U(n/k)}{a(n/k)}+\frac{1}{\gamma_0}\right\}+\gamma_0\frac{X_{n,n-k}-U(n/k)}{a(n/k)} \\ &:= II_1-II_2+II_3.\end{aligned}\tag{A.41}$$

By (2.8) and (A.40), we have that

$$\sqrt{k}II_1\xrightarrow{p}0\quad\text{and}\quad\sqrt{k}II_2\xrightarrow{p}0.\tag{A.42}$$

It follows from Drees et al. (2004) that

$$\sqrt{k}II_3\stackrel{d}{=}\gamma_0\sqrt{k}\frac{U(Y_{n,n-k})-U(n/k)}{a(n/k)}\xrightarrow{d}N(0,\gamma_0^2):=\gamma_0W^{*}.\tag{A.43}$$

Since $Y_{n,n-k}$ is independent of $\{Y_{n,n-i+1}/Y_{n,n-k}\}_{i=1}^{k}$, the theorem is proved by combining (2.13), (A.38), Theorems 1 and 2 and (A.41)–(A.43). □

Proof of Theorem 4. By writing

$$\frac{\hat{\sigma}}{a(n/k)} - 1 = \hat{\gamma}\left\{\frac{\hat{\tau}}{a(n/k)} + \frac{1}{\gamma_0}\right\} + (\hat{\gamma} - \gamma_0)/\gamma_0$$

and

$$\frac{\hat{\sigma}}{a(U^{-}(X_{n,n-k}))} - 1 = \hat{\gamma}\left\{\frac{\hat{\tau}}{a(U^{-}(X_{n,n-k}))} + \frac{1}{\gamma_0}\right\} + (\hat{\gamma} - \gamma_0)/\gamma_0,$$

the theorem follows from Theorem 3 and (A.38). □

Proofs of Theorems 5 and 6. Write

$$\frac{\hat{\theta} - \theta}{a(U^{-}(X_{n,n-k}))} = \frac{U(U^{-}(X_{n,n-k})) - U(\infty)}{a(U^{-}(X_{n,n-k}))} - \frac{1}{\gamma} + \frac{\hat{\tau}}{a(U^{-}(X_{n,n-k}))} + \frac{1}{\gamma}.$$

Then Theorems 5 and 6 follow from Theorems 1 and 2, (A.40), (2.8) and the fact that

$$a(U^{-}(X_{n,n-k}))/a(n/k) \xrightarrow{p} 1. \qquad \square$$

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