Iterated logarithm type behavior for weighted sums of i.i.d. random variables

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A R T I C L E   I N F O

Article history:
Received 22 May 2008
Received in revised form 6 October 2008
Accepted 8 October 2008
Available online 19 October 2008

MSC:
60G50
60F15
60G50

A B S T R A C T

For a sequence of i.i.d. mean 0 random variables \(\{X, X_n; n \geq 1\}\) with weighted partial sums \(S_n(X, w(\cdot)) = \sum_{k=1}^{n} w \left( \frac{k}{n} \right) X_k, n \geq 1\) where \(w(t), 0 \leq t \leq 1\) is a Lipschitz function of order 1 with \(\|w(\cdot)\|_2 = \int_0^1 w^2(t)dt > 0\), necessary and sufficient conditions are provided for \(X\) to enjoy iterated logarithm type behavior of the form
\[0 < \limsup_{n \to \infty} \frac{|S_n|}{\sqrt{n h(n)}} < \infty\]
almost surely where \(h(\cdot)\) is a positive, nondecreasing function which is slowly varying at infinity. Some corollaries are presented for particular choices of \(h(\cdot)\).

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1. Introduction

Throughout, \(\{X, X_n; n \geq 1\}\) is a sequence of real-valued independent and identically distributed (i.i.d.) random variables and, as usual, their partial sums will be denoted by \(S_n = \sum_{i=1}^{n} X_i, n \geq 1\). Let \(w(\cdot)\) be a continuous function on \([0, 1]\). Write
\[\|w(\cdot)\|_2 = \sqrt{\int_0^1 w^2(t)dt}\]
The weighted sums associated with \(w(\cdot)\) will be denoted by \(S_n(X, w(\cdot)) = \sum_{k=1}^{n} w \left( \frac{k}{n} \right) X_k, n \geq 1\). Of course, \(S_n(X, w(\cdot)) = S_n, n \geq 1\) if \(w(\cdot) \equiv 1\).

As in Einmahl and Li (2005), let \(\mathcal{H}\) denote the set of all continuous, nondecreasing functions \(h(\cdot) : [0, \infty) \to (0, \infty)\) with \(\lim_{t \to \infty} h(t) = \infty\) which are slowly varying at infinity. Set \(f_r(t) = \exp((Lt)^r), 0 \leq r \leq 1\). Here and throughout, \(Lx = \log_x(e \vee x)\) for \(x \geq 0\). Given \(0 \leq q < 1\), let \(\mathcal{H}_q\) denote the class of functions \(h(\cdot) \in \mathcal{H}\) such that
\[\lim_{t \to \infty} \frac{h(f_r(t))}{h(t)} = 1, \quad 0 < r < 1 - q\]
and set \(\mathcal{H}_1 = \mathcal{H}\). Examples of functions in \(\mathcal{H}_0\) are \(h(t) = (Lt)^r, r > 0\) and \(h(t) = (Lt)^p, p > 0\).

Einmahl and Li (2005) recently provided necessary and sufficient conditions for (two-sided) iterated logarithm type behavior of the form
\[0 < \limsup_{n \to \infty} \frac{S_n}{\sqrt{n h(n)}} = -\liminf_{n \to \infty} \frac{S_n}{\sqrt{n h(n)}} < \infty\]
almost surely (a.s.)

or
\[0 < \limsup_{n \to \infty} \frac{|S_n|}{\sqrt{n h(n)}} < \infty\] a.s.

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doi:10.1016/j.spl.2008.10.010
to prevail when \( h(\cdot) \) is from a suitable class of positive, continuous, nondecreasing functions with \( \lim_{t \to \infty} h(t) = \infty \) and which are slowly varying at infinity. Particular choices of \( h(\cdot) \) in (1.1) yield new analogues of the celebrated theorem of Hartman and Wintner (1941) when \( \mathbb{E}X = 0 \) and \( \mathbb{E}X^2 \) is finite or infinite.

Write
\[
\mathcal{L}_1 = \{ w(\cdot) ; \ w(t), \ 0 \leq t \leq 1 \text{ is a Lipschitz function of order 1} \}.
\]

This note is an extension of the work of Einmahl and Li (2005). We shall provide necessary and sufficient conditions for \( X \) to enjoy iterated logarithm type behavior of the weighted sum form
\[
0 < \limsup_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{\sqrt{nh(n)}} < \infty \text{ a.s.}
\]
when \( h(\cdot) \) is also from a suitable class of positive, continuous, nondecreasing functions with \( \lim_{t \to \infty} h(t) = \infty \) and which are slowly varying at infinity and \( w(\cdot) \in \mathcal{L}_1 \) with \( \|w(\cdot)\|_2 > 0 \). It is interesting to observe that the norming sequence \( \sqrt{nh(n)}, n \geq 1 \) does not depend on the choice of \( w(\cdot) \in \mathcal{L}_1 \) and that in general the almost sure value of
\[
\limsup_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{\sqrt{nh(n)}},
\]
depends on \( w(\cdot) \in \mathcal{L}_1 \) only through the value of \( \|w(\cdot)\|_2 \). The class \( \mathcal{L}_1 \) is large and contains, for example, every continuous function on \([0, 1] \) with a bounded derivative on \([0, 1] \).

The following is the main result of this paper.

**Theorem 1.** Let \( \{X, n ; n \geq 1 \} \) be a sequence of i.i.d. random variables. Given a function \( h(\cdot) \in \mathcal{H}_q \) where \( 0 \leq q \leq 1 \), set \( \Psi(x) = \sqrt{xh(x)} \), \( x \geq 0 \). Write \( H(x) = \mathbb{E}(X^21(|X| \leq x)), x \geq 0 \). If
\[
\mathbb{E}X = 0, \ \mathbb{E}\Psi^{-1}(|X|) < \infty, \text{ and } \limsup_{n \to \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} H(x) = \frac{\lambda^2}{2} \tag{1.2}
\]
where \( 0 \leq \lambda < \infty \), then for every \( w(\cdot) \in \mathcal{L}_1 \)
\[
(1 - q)^{1/2}\lambda \|w(\cdot)\|_2 \leq \limsup_{n \to \infty} \frac{S_n(X, w(\cdot))}{\sqrt{nh(n)}} \leq \lambda \|w(\cdot)\|_2 \text{ a.s.} \tag{1.3}
\]
and
\[
-\lambda \|w(\cdot)\|_2 \leq \liminf_{n \to \infty} \frac{S_n(X, w(\cdot))}{\sqrt{nh(n)}} \leq -(1 - q)^{1/2}\lambda \|w(\cdot)\|_2 \text{ a.s.} \tag{1.4}
\]
Conversely, if \( q < 1 \) and
\[
\limsup_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{\sqrt{nh(n)}} < \infty \text{ a.s.} \tag{1.5}
\]
for some \( w(\cdot) \in \mathcal{L}_1 \) with \( \|w(\cdot)\|_2 > 0 \), then
\[
\mathbb{E}\Psi^{-1}(|X|) < \infty \text{ and } \limsup_{n \to \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} H(x) < \infty. \tag{1.6}
\]
Moreover, if \( q < 1 \) and (1.5) holds for some \( w(\cdot) \in \mathcal{L}_1 \) with \( \int_0^1 w(t)dt \neq 0 \), then (1.2) holds where \( 0 \leq \lambda < \infty \).

The second condition of (1.2) is of course equivalent to
\[
\sum_{n=1}^{\infty} P \left(|X| \geq \sqrt{nh(n)} \right) < \infty.
\]

For a numerical sequence \( \{x_n; \ n \geq 1 \} \), we denote its cluster set (i.e., its set of limit points) by \( C \{x_n; \ n \geq 1 \} \). For functions \( h(\cdot) \in \mathcal{H}_0 \), we obtain a complete analogue of the Hartman and Wintner (1941) law of the iterated logarithm for weighted sums of i.i.d. random variables.

**Theorem 2.** Assume that \( h(\cdot) \in \mathcal{H}_0 \) and (1.2) holds. Then, for every \( w(\cdot) \in \mathcal{L}_1 \),
\[
\limsup_{n \to \infty} \frac{S_n(X, w(\cdot))}{\sqrt{nh(n)}} = (\pm)\lambda \|w(\cdot)\|_2 \text{ a.s.} \tag{1.7}
\]
and
\[
C\left( \frac{S_n(X, w(\cdot))}{\sqrt{nh(n)}} ; \ n \geq 1 \right) = [-\lambda \|w(\cdot)\|_2, \ \lambda \|w(\cdot)\|_2] \quad \text{a.s.}
\]

Conversely, if (1.7) holds for some constant \( 0 \leq \lambda < \infty \) and some \( w(\cdot) \in \mathcal{L}_1 \) with \( \|w(\cdot)\|_2 > 0 \), then (1.6) holds. Moreover, if (1.7) holds for some constant \( 0 \leq \lambda < \infty \) and some \( w(\cdot) \in \mathcal{L}_1 \) with \( \int_0^1 w(t)dt \neq 0 \), then (1.2) holds where \( \lambda \) is as in (1.7).

**Remark 1.** Theorem 2 is due to Li and Tomkins (1996, Theorem 2.1), for \( h(x) = 2Lx \).

Since Theorem 2 is just a special case of Theorem 1, we will only give the proof of Theorem 1. We will give the proofs of the first part (sufficiency) and the second part (necessity) of Theorem 1 in Sections 2 and 3, respectively.

### 2. Proof of Theorem 1: Sufficiency

**Proof of Theorem 1 (Sufficiency).** Let \( C[0, 1] \) denote the set of continuous, real-valued functions \( f(\cdot) \) on \( [0, 1] \). Write
\[
\mathcal{K} = \left\{ f(\cdot) \in C[0, 1] : f(t) = \int_0^t g(s)ds, \ 0 \leq t \leq 1 \right\}
\]
For \( n \geq 1 \), let \( S(n) \) denote the polygonal process of order \( n \); that is,
\[
S(n)(t) = S(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor)X_{\lfloor nt \rfloor + 1}, \quad 0 \leq t \leq 1
\]
where \( S_0 = 0 \) and \( \lfloor x \rfloor \) denotes the integer part of \( x \). Since \( w(\cdot) \in C[0, 1] \) is a Lipschitz function of order 1, one can see that there exists a constant \( 0 < c < \infty \), depending on \( w(\cdot) \) only, such that for all \( n \geq 1 \),
\[
\left| \sum_{k=1}^n \frac{w(k)}{n} X_k - \int_0^1 w(t)ds(n)(t) \right| \leq \frac{c}{n} \sum_{k=1}^n |X_k| \to_c cE|X| < \infty \quad \text{a.s.}
\]
by the first condition in (1.2) and the Kolmogorov strong law of large numbers. Thus
\[
\lim_{n \to \infty} \left| \frac{S_n(X, w(\cdot))}{\sqrt{nh(n)}} - \int_0^1 w(t)ds(n)(t) \right| = 0 \quad \text{a.s.} \tag{2.1}
\]
Write \( \psi(f) = w(1)f(1) - \int_0^1 f(t)dw(t) \). Clearly \( \psi : C[0, 1] \to \mathbb{R} = (-\infty, \infty) \) is a continuous function such that
\[
\psi \left( \frac{S(n)/\sqrt{nh(n)}}{\lambda} \right) = \int_0^1 \frac{w(t)ds(n)(t)}{\sqrt{nh(n)}}, \ n \geq 1.
\]
Thus, under (1.2), by Theorem 1 of Einmahl (2007) and Theorem 1 of Einmahl and Li (2005), we conclude that with probability 1,
\[
\left\{ \psi \left( \frac{S(n)/\sqrt{nh(n)}}{\lambda} \right) ; \ n \geq 1 \right\} \quad \text{is relatively compact in} \ \mathbb{R} \tag{2.2}
\]
and
\[
C \left( \left\{ \psi \left( \frac{S(n)/\sqrt{nh(n)}}{\lambda} \right) ; \ n \geq 1 \right\} \right) = \psi (\alpha_0 \mathcal{K}) \quad \text{a.s.} \tag{2.3}
\]
for some constant \( \alpha_0 \) such that \( (1-q)^{1/2}\lambda \leq \alpha_0 \leq \lambda \). We now show that
\[
\psi (\alpha_0 \mathcal{K}) = [-\alpha_0 \|w(\cdot)\|_2, \alpha_0 \|w(\cdot)\|_2]. \tag{2.4}
\]
In fact,
\[
\sup_{f \in \mathcal{K}} \psi (\alpha_0 f) = \alpha_0 \sup_{f \in \mathcal{K}} \left( w(1)f(1) - \int_0^1 f(t)dw(t) \right)
\]
\[
= \alpha_0 \sup_{f \in \mathcal{K}} \int_0^1 w(t)f'(t)dt
\]
\[
= \alpha_0 \|w(\cdot)\|_2
\]
since
\[
\int_0^1 w(t)f'(t)dt = \int_0^1 w(t)g(t)dt \\
\leq \left(\int_0^1 w^2(t)dt\right)^{1/2} \left(\int_0^1 g^2(t)dt\right)^{1/2} \\
\leq \|w(.)\|_2
\]
and, for \( f(t) = \int_0^t \frac{w(s)}{\|w(.)\|_2}ds \), \( 0 \leq t \leq 1 \),
\[
\int_0^1 w(t)f'(t)dt = \|w(.)\|_2.
\]
Similarly,
\[
\inf_{f \in \mathcal{K}} \psi (\alpha_0 f) = -\alpha_0 \|w(.)\|_2.
\]
Thus, (2.4) follows. Since \((1-q)^{1/2} \leq \alpha_0 \leq \lambda \), the validity of (1.3) now follows immediately from (2.1)–(2.4). The conclusion (1.4) is then obtained from (1.3) by replacing \( \{X, X_n; n \geq 1\} \) by \( \{-X, -X_n; n \geq 1\} \).

\( \square \)

3. Proof of Theorem 1: Necessity

Set \( a_n = \sqrt{nh(n)} \), \( n \geq 1 \) and \( f_\epsilon(x) = \exp((Lx)^\epsilon) \), \( x > 1 \).

We will first present some known results which will be used in the proof of the necessity part of Theorem 1.

**Lemma 1** (Chow and Lai, 1973, Lemma 3). Let \( \{Y_n; n \geq 1\} \) and \( \{Z_n; n \geq 1\} \) be sequences of random variables such that \( Y_n + Z_n \to 0 \) a.s. Assume that \( \{F_n; n \geq 1\} \) is a monotone increasing sequence of \( \sigma \)-fields. For each \( n \geq 1 \), suppose that \( Y_1, \ldots, Y_n \) are adapted to \( F_n \), and that \( Z_n \) and \( F_n \) are independent. If \( Z_n \to_\mathbb{P} 0 \), then \( Y_n \to 0 \) a.s. and \( Z_n \to 0 \) a.s.

**Lemma 2** (Einmahl, 2007, Theorem 2). Let \( X \) be a mean zero random variable such that
\[
\sum_{n=1}^\infty \mathbb{P}(|X| > a_n) < \infty.
\]
If the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is rich enough, then one can define a sequence \( \{X_n; n \geq 1\} \) of independent copies of \( X \) and a standard Brownian motion \( \{W(t); t \geq 0\} \) such that as \( n \to \infty \),
\[
\sup_{0 \leq t \leq 1} |S_{(n)}(t) - \sigma_n W_{(n)}(t)| = o(a_n) \quad \text{a.s.},
\]
where \( W_{(n)}(t) = W(nt), \ 0 \leq t \leq 1 \) and \( \sigma_n^2 = H(a_n), \ n \geq 1 \).

**Lemma 3** (Pruitt, 1981, Lemma 3.4). Let \( \{A_n; n \geq 1\}, \{B_n; n \geq 1\} \) be sequences of events such that the events \( A_n, n \geq 1 \) are independent and for each \( n \geq 1 \), the pair \( A_n \) and \( B_n \) are independent. Suppose that \( \sum_{n=1}^\infty \mathbb{P}(A_n) = \infty \) and \( \mathbb{P}(B_n) > c > 0 \) for all \( n \geq 1 \). Then \( \mathbb{P}(A_n \cap B_n \text{ i.o.}) > 0 \).

**Proof of Theorem 1 (Necessity).** As will be apparent from the argument below, we only need to show that \( 0 \leq q < 1 \) and (1.5) holding for some \( w(.) \in \mathcal{L}_1 \) with \( \int_0^1 w(t)dt \neq 0 \) imply that (1.2) holds where \( 0 \leq \lambda < \infty \).

First we will show that \( \mathbb{E} \Psi^{-1}(|X|) < \infty \), which is equivalent to
\[
\sum_{n=1}^\infty \mathbb{P}(|X| > \varepsilon a_n) < \infty
\]
for all \( \varepsilon > 0 \) (or for some \( \varepsilon > 0 \)). In other words, \( \mathbb{E} \Psi^{-1}(|X|) = \infty \) is equivalent to
\[
\sum_{n=1}^\infty \mathbb{P}(|X| > \varepsilon a_n) = \infty \quad \text{for all } \varepsilon > 0,
\]
which implies there exists a positive function \( b(x) \not\to \infty \) as \( x \to \infty \) such that
\[
\sum_{n=1}^\infty \mathbb{P}(|X| > b(n)a_n) = \infty.
\] (3.1)
We will prove that this results in a contradiction if \( \mathbb{E}\Psi^{-1}(|X|) = \infty \).

Now from (1.4), we have
\[
\lim_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{b(\sqrt{n})a_n} = 0 \quad \text{a.s.}
\]
Let \( \{X', X'_n; n \geq 1\} \) be an independent copy of \( \{X, X_n; n \geq 1\} \) and set \( X' = X - X' \) and \( X'_n = X_n - X'_n \) for \( n \geq 1 \). Then the above relation gives
\[
\lim_{n \to \infty} \frac{|S_n(X', w(\cdot))|}{b(\sqrt{n})a_n} = 0 \quad \text{a.s.}
\]
and thus
\[
\lim_{n \to \infty} \frac{|S_n(X', w(\cdot))|}{b(\sqrt{n})a_n} = 0 \quad \text{a.s.} \tag{3.2}
\]
Since \( w(t), 0 \leq t \leq 1 \) is continuous with \( \|w(\cdot)\|_2 > 0 \), there exist a \( t_0 \in (0, 1) \) and a constant \( c > 0 \) such that \( |w(t)| > c > 0 \) in a neighborhood of \( t_0 \). Let \( n(j) = [j/t_0] \) and \( m(j) = j \). Then
\[
\frac{|w(\frac{m(j)}{n(j)})|}{c} \geq c > 0 \quad \text{for all large } j \tag{3.3}
\]
Set for all large \( j \)
\[
Y_j = \sum_{k=1}^{m(j)} w\left(\frac{k}{n(j)}\right) X^s_k
\]
and
\[
Z_j = \sum_{k=m(j)+1}^{n(j)} w\left(\frac{k}{n(j)}\right) X^s_k
\]
Clearly it follows from (3.2) that
\[
Y_j + Z_j \to 0 \quad \text{a.s. as } j \to \infty.
\]
Since \( Y_j \) and \( Z_j \) are independent symmetric random variables, it follows from Lévy’s inequality that for all \( \delta > 0 \),
\[
\mathbb{P}(|Z_j| > \delta) \leq 2\mathbb{P}(|Y_j + Z_j| > \delta) \to 0 \quad \text{as } j \to \infty.
\]
Thus \( Z_j \to_{\mathbb{P}} 0 \). Let \( \mathcal{F}_j \) be the \( \sigma \)-field generated by the random variables \( X_i, 1 \leq i \leq m(j) \). By Lemma 1, we have
\[
Y_j = \frac{\sum_{k=1}^{m(j)} w\left(\frac{k}{n(j)}\right) X^s_k}{b(\sqrt{n(j)})a_{m(j)}} \to 0 \quad \text{a.s.} \tag{3.4}
\]
Similarly, one can show that
\[
\frac{\sum_{k=1}^{m(j)-1} w\left(\frac{k}{n(j)}\right) X^s_k}{b(\sqrt{n(j)})a_{m(j)}} \to 0 \quad \text{a.s.} \tag{3.5}
\]
From (3.4) and (3.5) we have
\[
\frac{w\left(\frac{m(j)}{n(j)}\right) X^s_{m(j)}}{b(\sqrt{n(j)})a_{m(j)}} \to 0 \quad \text{a.s.}
\]
which implies by (3.3) that
\[
\frac{X^s_{m(j)}}{b(\sqrt{n(j)})a_{m(j)}} \to 0 \quad \text{a.s.}
\]
Then by the Borel-Cantelli lemma,
\[ \sum_{j=1}^{\infty} \mathbb{P} \left( |X^j| > 0.5b(j)a_j \right) < \infty. \]

Let \( \mu(X) \) denote a median of the random variable \( X \). Note that
\[ \mathbb{P}(|X - \mu(X)| > x) \leq 2\mathbb{P} \left( |X^j| > x \right) \quad \text{for all } x > 0 \]
(cf., e.g., Chow and Teicher (1997), Lemma 10.1.1). We thus have
\[ \sum_{j=1}^{\infty} \mathbb{P} \left( |X - \mu(X)| > 0.5b(j)a_j \right) < \infty \]
implying
\[ \sum_{j=1}^{\infty} \mathbb{P} \left( |X| > b(j)a_j \right) < \infty \]
which contradicts (3.1). Consequently, \( \mathbb{E} \Psi^{-1}(|X|) < \infty \).

Let \( \{a_{n,k} \}, 1 \leq k \leq n, n \geq 1 \) be a triangular array of real numbers. Under the assumption that \( \mathbb{E}|X| < \infty \) and
\[ \sup_{1 \leq k \leq n, n \geq 1} |a_{n,k}| < \infty, \]
Choi and Sung (1987) showed that
\[ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} a_{n,k} (X_k - \mathbb{E}X)}{n} = 0 \quad \text{a.s.} \]
Also note that \( \mathbb{E} \Psi^{-1}(|X|) < \infty \) ensures that \( \mathbb{E}|X| < \infty \). Thus from the given conditions we can conclude that
\[ 0 = \limsup_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{n} \geq \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} w \left( \frac{k}{n} \right) \mathbb{E}X}{n} - \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} w \left( \frac{k}{n} \right) (X_k - \mathbb{E}X)}{n} \]
thereby yielding \( \mathbb{E}X = 0. \)

Finally, we proceed to prove the third part in (1.2). Otherwise, we assume that
\[ \limsup_{x \to \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} H(x) = \infty, \quad \text{(3.6)} \]
and show that this yields the following contradiction:
\[ \limsup_{n \to \infty} \frac{|S_n(X, w(\cdot))|}{a_n} = \infty \quad \text{a.s.} \]
which will follow provided we are able to show that
\[ \limsup_{n \to \infty} \frac{\sigma_n \left| \sum_{k=1}^{n} w \left( \frac{k}{n} \right) V_k \right|}{a_n} = \infty \quad \text{a.s.,} \quad \text{(3.7)} \]
where \( V_k = W(k) - W(k - 1) \) since, by Lemma 2, \( \mathbb{E} \Psi^{-1}(|X|) < \infty \) implies that
\[ \left| S_n(X, w(\cdot)) - \sigma_n \sum_{k=1}^{n} w \left( \frac{k}{n} \right) V_k \right| = \left| \sum_{k=1}^{n} w \left( \frac{k}{n} \right) \left( S_k - \sigma_n W_k \right) \right| \\
= \left| \sum_{k=1}^{n-1} \left( w \left( \frac{k}{n} \right) - w \left( \frac{k+1}{n} \right) \right) \left( S_k - \sigma_n W_k \right) + w(1) \left( S_n - \sigma_n W_n \right) \right| \\
\leq (c + |w(1)|) \sup_{0 \leq t \leq 1} \left| S_n(t) - \sigma_n W_n(t) \right| \\
= o \left( a_n \right) \quad \text{a.s.} \]
where $0 < c < \infty$ is a constant, depending on $w(\cdot)$ only, such that $|w(x) - w(y)| \leq c|x - y|$ for all $x, y$ in $[0, 1]$.

Since $h(\cdot) \in \mathcal{H}_q$ for some $q \in (0, 1)$, it is easy to show that $\Psi^{-1}(xLLx) \sim \Psi^{-1}(aLLx)^2$ as $x \to \infty$. We thus have from (3.6) that

$$
\infty = \limsup_{x \to \infty} \frac{\Psi^{-1}(xLLx)}{x^2} H(x) = \limsup_{n \to \infty} \frac{nLLn}{a_n^2} H(a_n).
$$

Choose a sequence of integers $\{n_k; \ k \geq 1\}$ such that $n_{k+1} > n_k^2$ for all $k \geq 1$ and

$$
\lim_{k \to \infty} \frac{n_kLLn_k}{a_{n_k}^2} H(a_{n_k}) = \infty.
$$

For $0 < \tau < 1 - q$, let $\mathcal{N}_\tau = \{j; \ n_k \leq j \leq n_kf_r(n_k)\}$ for some $k \geq 1$. For $n_k \leq j \leq n_kf_r(n_k)$ we have

$$
\frac{j}{a_j^2} = \frac{1}{h(j)} \geq \frac{1}{h(n_kf_r(n_k))} \sim \frac{1}{h(n_k)} = \frac{n_k}{a_{n_k}^2},
$$

and hence

$$
\lim_{\mathcal{N}_\tau \not\to \infty} \frac{(jLLj)\sigma_j^2}{a_j^2} = \lim_{\mathcal{N}_\tau \not\to \infty} \frac{jLLj}{a_j^2} H(a_j) \geq \lim_{k \to \infty} \frac{n_kLLn_k}{a_{n_k}^2} H(a_{n_k}) = \infty.
$$

From this, (3.7) will follow if we are able to show that

$$
\limsup_{\mathcal{N}_\tau \not\to \infty} \frac{\sum_{k=1}^{j} w\left(\frac{j}{n}\right)V_k}{\sqrt{jLLj}} > 0 \quad \text{a.s.} \tag{3.8}
$$

To this end, let $s_0 \in (0, 1)$ be a number such that $c_0 = (1/2) \int_{s_0}^1 u^2(t)dt > 0$. For each $n > 1/s_0$, define

$$
T_n = \sum_{1 \leq k < n_0} w\left(\frac{k}{n}\right)V_k, \quad U_n = \sum_{n_0 \leq k \leq n} w\left(\frac{k}{n}\right)V_k.
$$

Then $T_n$ and $U_n$ are independent and normally distributed. It is easy to see that $\mathbb{E}U_n = 0$ and $\mathbb{E}U_n^2 = \sum_{n_0 \leq k \leq n} w^2\left(\frac{k}{n}\right) \geq 1.5n_0$ for all large $n$, and thus

$$
P\left( U_n > c_0 \tau^{1/2} \sqrt{nLLn} \right) \geq \exp \left\{-\frac{\tau LLn}{2}\right\} = (Ln)^{-\tau/2}.
$$

Define $m_i = 2^i, \ i \geq 1$ and let $\mathcal{N}_{\tau}^m = \{m_i; \ m_i \in \mathcal{N}_{\tau}\}$. For each $j \in \mathcal{N}_{\tau}^m$, define

$$
A_j = \left\{ U_j > c_0 \tau^{1/2} \sqrt{jLLj} \right\} \quad \text{and} \quad B_j = \left\{ T_j \geq 0 \right\}.
$$

Obviously, $P(B_j) = 1/2$. $A_j$ and $B_j$ are independent for each $j$, and the $A_j, \ j \in \mathcal{N}_{\tau}^m$ are independent. Now for some constants $C_1, \ C_2$ in $(0, \infty)$ and all $k \geq 1$,

$$
\# \{m_i; \ n_k \leq m_i \leq n_kf_r(n_k)\} \geq C_1 (Ln_k)^\tau
$$

and

$$
Ln_k \leq C_2 Ln_k \quad \text{whenever} \ m_i \leq n_kf_r(n_k).
$$

Hence

$$
\sum_{j \in \mathcal{N}_{\tau}^m} P(A_j) \geq \sum_{k=1}^{\infty} \sum_{n_k \leq m_i \leq n_kf_r(n_k)} (Ln_k)^{-\tau/2} \geq C_1C_2^{-\tau/2} \sum_{k=1}^{\infty} (Ln_k)^\tau/2 = \infty.
$$

Applying Lemma 3 yields

$$
P\left( \limsup_{\mathcal{N}_\tau \not\to \infty} \frac{\sum_{k=1}^{j} w\left(\frac{j}{n}\right)V_k}{\sqrt{jLLj}} \geq c_0 \tau^{1/2} \right) \geq P (A_j \cap B_j \ i.o \ in \ \mathcal{N}_{\tau}^m) > 0,
$$

and (3.8) then follows from the Kolmogorov $0 - 1$ law. This completes the proof. □
4. Two corollaries

Theorem 2 can be illustrated by considering the following two special cases:

Case I. Choose $h(x) = 2(LLx)^p$ where $p \geq 1$. Then one can check that

$$\lim_{x \to \infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{1/(2(LLx)^{p-1})} = 1.$$\[4.2\]

Case II. Take $h(x) = 2(Lx)^r$ where $r > 0$. Then one easily sees that

$$\lim_{x \to \infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{LLx/(Lx)^r} = 2^{-(r+1)}.$$\[4.4\]

Thus, Theorem 2 yields the following two corollaries.

**Corollary 1.** Let $p \geq 1$. If

$$\text{E}X = 0, \quad \text{E} \left( \frac{X^2}{(LLX)^p} \right) < \infty, \quad \text{and} \quad \lim_{x \to \infty} \sup (LLx)^{1-p}H(x) = \lambda^2$$ \[4.1\]

where $0 \leq \lambda < \infty$, then for every $w(\cdot) \in \mathcal{L}_1$

$$\lim_{n \to \infty} \sup (\lim \inf) \frac{S_n(X, w(\cdot))}{\sqrt{2n(LL)^2}} = (\pm)\lambda \|w(\cdot)\|_2 \quad \text{a.s.} \quad \text{and}$$ \[4.2\]

$$\mathcal{C} \left( \frac{S_n(X, w(\cdot))}{\sqrt{2n(LL)^2}}, \quad n \geq 1 \right) = [-\lambda \|w(\cdot)\|_2, \lambda \|w(\cdot)\|_2] \quad \text{a.s.}$$

Conversely, if (4.2) holds for some constant $0 \leq \lambda < \infty$ and some $w(\cdot) \in \mathcal{L}_1$ with $\|w(\cdot)\|_2 > 0$, then

$$\text{E} \left( \frac{X^2}{(LLX)^p} \right) < \infty \quad \text{and} \quad \lim_{x \to \infty} \sup (LLx)^{1-p}H(x) < \infty.$$ \[4.3\]

Moreover, if (4.2) holds for some constant $0 \leq \lambda < \infty$ and some $w(\cdot) \in \mathcal{L}_1$ with $\int_0^1 w(t)dt \neq 0$, then (4.1) holds where $\lambda$ is as in (4.2).

**Remark 2.** Li et al. (2008) recently showed that the second condition of (4.3) implies the first condition of (4.3). Consequently, the first part of Corollary 1 holds if the condition (4.1) is replaced by

$$\text{E}X = 0 \quad \text{and} \quad \lim_{x \to \infty} \sup (LLx)^{1-p}H(x) = \lambda^2.$$ \[4.4\]

**Remark 3.** The choice $w(\cdot) \equiv 1, p = 1$, and $0 < \lambda < \infty$ in Corollary 1 yields the classical Hartman and Wintner (1941) LIL and its converse due to Strassen (1966).

**Corollary 2.** Let $r > 0$. If

$$\text{E}X = 0, \quad \text{E} \left( \frac{X^2}{(LLX)^r} \right) < \infty, \quad \text{and} \quad \lim_{x \to \infty} \frac{LLx}{2^r(Lx)^r}H(x) = \lambda^2$$ \[4.4\]

where $0 \leq \lambda < \infty$, then for every $w(\cdot) \in \mathcal{L}_1$

$$\lim_{n \to \infty} \sup (\lim \inf) \frac{S_n(X, w(\cdot))}{\sqrt{2n(L)^2}} = (\pm)\lambda \|w(\cdot)\|_2 \quad \text{a.s.} \quad \text{and}$$ \[4.5\]

$$\mathcal{C} \left( \frac{S_n(X, w(\cdot))}{\sqrt{2n(L)^2}}, \quad n \geq 1 \right) = [-\lambda \|w(\cdot)\|_2, \lambda \|w(\cdot)\|_2] \quad \text{a.s.}$$

Conversely, if (4.5) holds for some constant $0 \leq \lambda < \infty$ and some $w(\cdot) \in \mathcal{L}_1$ with $\|w(\cdot)\|_2 > 0$, then

$$\text{E} \left( \frac{X^2}{(LLX)^r} \right) < \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{LLx}{2^r(Lx)^r}H(x) < \infty.$$ \[4.6\]

Moreover, if (4.5) holds for some constant $0 \leq \lambda < \infty$ and some $w(\cdot) \in \mathcal{L}_1$ with $\int_0^1 w(t)dt \neq 0$, then (4.4) holds where $\lambda$ is as in (4.5).
Acknowledgments

The authors are grateful to the referee for carefully reading the manuscript and for pointing out to them obscureness and gaps in some of the arguments in the original version. The authors are also grateful to Professor Uwe Einmahl for his interest in their work and for his comments on a preliminary version of Theorem 1. The perceptive comments of the referee and Professor Einmahl enabled the authors to substantially improve the paper. The research of Deli Li was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada and the research of Yongcheng Qi was partially supported by NSF Grant DMS-0604176.

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