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The convergence rate for the normal approximation of extreme sums

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ABSTRACT

This paper derives the normal convergence rate for sums of extreme values. The exact rates are obtained under some regularity and smoothing conditions on the tail of the distribution of the sample.

1. Introduction and main results

Let $\{X, X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed random variables with distribution F, and for each $n \geq 1$ let $X_{n,1} \leq \cdots \leq X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$. Define $S_{n,k} = \sum_{j=1}^{k} X_{n,n-j+1}$ as an extreme sum, where $k = k_n$ satisfies

$$\lim_{n\to\infty} k_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{k_n}{n} = 0. \tag{1}$$

The extreme sums have been widely studied in the literature. On one hand, the study is motivated by the interest of how the asymptotic properties of partial sums $\sum_{j=1}^{n} X_j$ are influenced by the extreme sums. On the other hand, these extreme observations play an important role in estimating the tail of a distribution in extreme-value statistics.

Assume that F belongs to the domain of attraction of an extreme value distribution, i.e., there exist constants $c_n > 0$ and $d_n \in R$ such that

$$\frac{X_{n,n} - d_n}{c_n} \xrightarrow{d} \exp\{-(1+\gamma x)^{-1/\gamma}\} =: G_\gamma(x) \tag{2}$$

for some $\gamma \in R$. It is known that the sums $\{S_{n,k_n}\}$, after being suitably centered and normalized, have a limiting distribution. See, e.g., [1–5]. In particular, Csörgö, Haeusler and Mason [5] proved that the limiting distribution of $\{S_{n,k_n}\}$ is normal if and only if $\gamma \leq 1/2$.

Denote $\omega_F = \sup\{x : F(x) < 1\}$. It is well known that the following von-Mises conditions are sufficient for (2) to hold: F has the derivative function F' satisfying

$$\gamma > 0 : \omega_F = \infty, \quad \lim_{x\to\infty} \frac{xF'(x)}{1-F(x)} = \frac{1}{\gamma};$$

$$\gamma < 0 : \omega_F < \infty, \quad \lim_{x\to\infty} \frac{(\omega_F - x)F'(x)}{1-F(x)} = -\frac{1}{\gamma};$$

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$$\gamma = 0: \quad \lim_{x\to\omega_F} \frac{F'(x)}{[1-F(x)]^2}\int_x^{\omega_F}[1-F(u)]\mathrm{d}u = 1.$$

Let $V(t) = \inf\{x : [1-F(x)]^{-1} \ge t\}$ for $t \ge 1$. Then these von-Mises conditions are equivalent to the assumption that V has a derivative function $V' \in Rv(\gamma-1)$, where $Rv(\gamma-1)$ denotes the class of all regularly varying functions with index $\gamma-1$, i.e.,

$$\lim_{t\to\infty}\frac{V'(tx)}{V'(t)} = x^{\gamma-1}, \quad \forall x > 0. \tag{3}$$

See [6] for more details.

The main objective of this paper is to estimate rates of convergence to normality for the extreme sums under condition (1). Throughout the paper, Φ and ϕ denote the cumulative standard normal distribution function and its density, respectively, D^2X denotes the variance of a random variable X, and Z denotes a random variable with the distribution function $F_Z(x) = 1-x^{-1}, x \ge 1$. Moreover, we shall let

$$a_n = \frac{n-k_n+1}{n+1}; \qquad b_n = \sqrt{\frac{a_n(1-a_n)}{n+1}};$$
$$l_n = \frac{1}{1-a_n}; \quad \tau_n = \frac{\sqrt{a_n}\,(EV(l_nZ)-V(l_n))}{DV(l_nZ)};$$
$$\Delta_n(x) = P\left(\frac{S_{n,k_n}-k_nEV(l_nZ)}{\sqrt{k_n(1+\tau_n^2)D^2V(l_nZ)}} \le x\right) - \Phi(x).$$

The main results are stated in Theorems 1 and 2.

Theorem 1. *Under (1), if (3) holds for some $\gamma \in [1/3, 1/2)$, then*

$$\sup_{x\in R}|\Delta_n(x)| = o(k_n^{-\epsilon}) \tag{4}$$

holds for any

$$\epsilon \in (0, (2\gamma)^{-1}-1). \tag{5}$$

Theorem 2. *Under (1), if (3) holds for $\gamma < 1/3$, then*

$$\Delta_n(x) = \frac{1}{\sqrt{k_n}}\frac{2(1-2\gamma)^{1/2}}{6\,(2(1-\gamma))^{3/2}\,(1-3\gamma)}\left(\left(\xi_1(\gamma)+\xi_2(\gamma)x^2\right)\phi(x)+\Phi_\gamma(x)\right)+o\left(\frac{1}{\sqrt{k_n}}\right) \tag{6}$$

holds uniformly on R, where

$$\xi_1(\gamma) = 7-8\gamma+9\gamma^2, \qquad \xi_2(\gamma) = 1-16\gamma-15\gamma^2+26\gamma^3$$

and

$$\Phi_\gamma(x) = 2(1-2\gamma)(1-3\gamma)\int_{-\infty}^x(3u-u^3)\phi(u)\mathrm{d}u.$$

Since the proofs of the theorems are lengthy, we will split the proofs into a series of lemmas which are given in the next two sections.

2. Proof of Theorem 1

The following two lemmas are related to condition (3).

Lemma 1. *If (3) holds for $\gamma \in R$, then*

$$\lim_{t\to\infty}\frac{V(tx)-V(t)}{tV'(t)} = \frac{x^\gamma-1}{\gamma}, \quad \forall x > 0 \tag{7}$$

with convention $\frac{x^\gamma-1}{\gamma}|_{\gamma=0} = \log x$ for $x > 0$. Furthermore, for any $\delta > 0$, there exists a $t_\delta > 0$ such that

$$(1-\delta)\frac{x^{\gamma-\delta}-1}{\gamma-\delta} \le \frac{V(tx)-V(t)}{tV'(t)} \le (1+\delta)\frac{x^{\gamma+\delta}-1}{\gamma+\delta} \tag{8}$$

holds for all $x \ge 1$ and $t \ge t_\delta$.

Proof. Note that

$$\frac{V(tx)-V(t)}{tV'(t)}=\int_1^x \frac{V'(tu)}{V'(t)}\mathrm{d}u, \quad \forall x>0. \tag{9}$$

By the properties of regularly varying functions, the convergence in (3) is locally uniform. Therefore, we have

$$\lim_{t\to\infty}\frac{V(tx)-V(t)}{tV'(t)}=\int_1^x u^{\gamma-1}\mathrm{d}u=\frac{x^\gamma-1}{\gamma},$$

which is conclusion (7). Next, it follows from (3) that for any $\delta>0$, there exists $t_\delta>0$ such that

$$(1-\delta)x^{\gamma-\delta-1}\le\frac{V'(tx)}{V'(t)}\le(1+\delta)x^{\gamma+\delta-1}$$

for all $t\ge t_\delta$ and $x\ge 1$ (cf. [7]). By using (9), conclusion (8) follows which completes the proof. □

Lemma 2. *Conditions (1) and (3) imply*

$$\lim_{t\to\infty}\int_1^\infty\left[\frac{V(tv)-V(t)}{tV'(t)}\right]^j\frac{\mathrm{d}v}{v^2}=\int_1^\infty\frac{(v^\gamma-1)^j}{\gamma^j v^2}\mathrm{d}v, \quad \forall\gamma<j^{-1} \tag{10}$$

for all $j=1,2,\ldots$.

Proof. The lemma follows from (7), (8) and the dominated convergence theorem. □

From (10) with $j=1$, it follows that

$$\lim_{n\to\infty}\frac{EV(l_nZ)-V(l_n)}{l_nV'(l_n)}=\int_1^\infty\frac{v^\gamma-1}{\gamma}\frac{\mathrm{d}v}{v^2}=\frac{1}{1-\gamma}\quad\text{if }\gamma<1. \tag{11}$$

Moreover, by setting $j=1$ and $j=2$ in (10) we conclude that

$$\begin{aligned}\frac{D^2V(l_nZ)}{[l_nV'(l_n)]^2} &= \int_1^\infty\left[\frac{V(l_nv)}{l_nV'(l_n)}\right]^2\frac{\mathrm{d}v}{v^2}-\left\{\int_1^\infty\frac{V(l_nv)}{l_nV'(l_n)}\frac{\mathrm{d}v}{v^2}\right\}^2\\ &= \int_1^\infty\left[\frac{V(l_nv)-V(l_n)}{l_nV'(l_n)}\right]^2\frac{\mathrm{d}v}{v^2}-\left\{\int_1^\infty\left[\frac{V(l_nv)-V(l_n)}{l_nV'(l_n)}\right]\frac{\mathrm{d}v}{v^2}\right\}^2\\ &\to \int_1^\infty\left(\frac{v^\gamma-1}{\gamma}\right)^2\frac{\mathrm{d}v}{v^2}-\left(\int_1^\infty\frac{v^\gamma-1}{\gamma}\frac{\mathrm{d}v}{v^2}\right)^2\\ &= \frac{1}{(1-\gamma)^2(1-2\gamma)}\end{aligned}$$

as $n\to\infty$ for $\gamma<1/2$. Therefore

$$\lim_{n\to\infty}\frac{DV(l_nZ)}{l_nV'(l_n)}=\frac{1}{(1-\gamma)(1-2\gamma)^{1/2}}=:\sigma(\gamma)\quad\text{if }\gamma<1/2. \tag{12}$$

Let $Z_1,Z_2,\ldots$ be independent random variables having the same distribution function as Z and let $Z_{n,1}\le\cdots\le Z_{n,n}$ be the order statistics of $Z_1,\ldots,Z_n$. It is easily seen that

$$(X_{n,1},\ldots,X_{n,n})\overset{d}{=}(V(Z_{n,1}),\ldots,V(Z_{n,n})).$$

For every $u\in R$ set

$$a_n(u)=\{(a_n+b_nu)\wedge 1\}\vee 0;\qquad l_n(u)=(1-a_n(u))^{-1};$$

$$S_{n,k_n}(u)=\sum_{j=1}^{k_n}V(l_n(u)Z_j);\qquad \phi_n(u)=\phi(u)\left(1-\frac{(u^3-3u)}{3\sqrt{k_n}}\right).$$

Lemma 3. *Under (1), we have*

$$\sup_{x\in R}\left|P(S_{n,k_n}\le x)-\int_{-\infty}^\infty P(S_{n,k_n}(u)\le x)\phi_n(u)\mathrm{d}u\right|=o\left(\frac{1}{\sqrt{k_n}}\right). \tag{13}$$

Proof. Let $U_1,U_2,\ldots$ be independent random variables uniformly distributed over $(0,1)$ and $U_{n,1}\le\cdots\le U_{n,n}$ be the order statistics of $U_1,\ldots,U_n$. According to Cheng [8], we have

$$P(S_{n,k_n} \le x) = \int_0^1 P\left(\sum_{j=1}^{k_n} V((1-s)^{-1}Z_j) \le x\right) p_n(s)\mathrm{d}s,$$

where p_n denotes the density function of $U_{n,n-k_n+1}$, i.e.,

$$p_n(s) = \frac{n!}{k_n!(n-k_n-1)!} s^{n-k_n-1}(1-s)^{k_n}$$

if $s \in (0,1)$ and $p_n(s) = 0$ otherwise. Since the density function of $\hat{U}_{n,n-k_n+1} = (U_{n,n-k_n+1} - a_n)/b_n$ is $\hat{p}_n(u) = b_n p_n(a_n + b_n u)$, $\forall u \in R$, it follows that

$$P(S_{n,k_n} \le x) = \int_{-\infty}^{\infty} P\left(\sum_{j=1}^{k_n} V(l_n(u)Z_j) \le x\right) \hat{p}_n(u)\mathrm{d}u.$$

Note that we have under (1) that

$$\hat{p}_n(u) = \phi_n(u) + o\left(\frac{1}{\sqrt{k_n}}\right)\phi(u)(1+u^6)$$

for all $|u| \le k_n^{1/6}/2$. See, e.g., Theorem 4.7.1 in [9]. Therefore, (13) holds. □

Denote

$$T_{n,j}(u) = E[V(l_n(u)Z) - V(l_n)]^j - E[V(l_nZ) - V(l_n)]^j, \quad \text{for } j = 1, 2, \ldots$$

and let $d_n(u) = l_n(u)/l_n$, $\forall u \in R$. It is easily seen that for any $\epsilon_n = o(\sqrt{k_n})$,

$$\sup_{|u|\le\epsilon_n} |d_n(u) - 1| \sim \sup_{|u|\le\epsilon_n} |d_n^{-1}(u) - 1| \sim \frac{\epsilon_n}{\sqrt{k_n}} \to 0. \tag{14}$$

Lemma 4. *For any integer $j \ge 1$, if $\gamma < j^{-1}$, then*

$$\frac{T_{n,j}(u)}{[l_nV'(l_n)]^j} = \frac{ju[1+o(1)]}{\sqrt{k_n}} \int_1^{\infty} t^{\gamma-2}\left(\frac{t^\gamma - 1}{\gamma}\right)^{j-1} \mathrm{d}t \tag{15}$$

holds uniformly on $[-k_n^{1/6}, k_n^{1/6}]$.

Proof. We shall only show that (15) holds uniformly on $u \in [0, k_n^{1/6}]$ since the proof for $u \in [-k_n^{1/6}, 0]$ is similar. Note that

$$\begin{aligned} T_{n,j}(u) &= \int_1^{\infty} \frac{[V(l_n(u)s) - V(l_n)]^j - [V(l_ns) - V(l_n)]^j}{s^2}\mathrm{d}s \\ &= j\int_1^{\infty} \frac{\mathrm{d}s}{s^2} \int_{l_ns}^{l_n(u)s} [V(v) - V(l_n)]^{j-1}V'(v)\mathrm{d}v \\ &= jl_n \int_1^{\infty} \frac{\mathrm{d}s}{s} \int_1^{d_n(u)} [V(l_nsv) - V(l_n)]^{j-1}V'(l_nsv)\mathrm{d}v \\ &= jl_n \int_1^{d_n(u)} \mathrm{d}v \left\{\int_1^{\infty} - \int_1^{v}\right\} \frac{[V(l_nt) - V(l_n)]^{j-1}V'(l_nt)}{t}\mathrm{d}t \\ &=: T_{n,j}^{(1)}(u) + T_{n,j}^{(2)}(u). \end{aligned}$$

By (8), the Potter bounds for regular variation (cf., [10]) and the dominated convergence theorem, we have

$$\begin{aligned} \frac{T_{n,j}^{(1)}(u)}{[l_nV'(l_n)]^j} &= j[d_n(u) - 1]\int_1^{\infty} \left[\frac{V(l_nt) - V(l_n)}{l_nV'(l_n)}\right]^{j-1} \frac{V'(l_nt)}{V'(l_n)}\frac{\mathrm{d}t}{t} \\ &= \frac{ju[1+o(1)]}{\sqrt{k_n}} \int_1^{\infty} t^{\gamma-2}\left(\frac{t^\gamma - 1}{\gamma}\right)^{j-1}\mathrm{d}t \end{aligned}$$

uniformly on $[0, k_n^{1/6}]$. Moreover, it follows from (8), (14) and the Potter bounds that

$$\sup_{v\in(1,d_n(u))} \left|\int_1^{v} \left[\frac{V(l_nt) - V(l_n)}{l_nV'(l_n)}\right]^{j-1} \frac{V'(l_nt)}{V'(l_n)}\frac{\mathrm{d}t}{t}\right| \to 0$$

and therefore

$$\frac{T_{n,j}^{(2)}(u)}{[l_n V'(l_n)]^j} = o\left(\frac{1}{\sqrt{k_n}}\right)$$

holds uniformly on $u \in [0, k_n^{1/6}]$ as $n \to \infty$. The proof of the lemma is now completed. □

For simplicity, let C denote a positive constant which may differ as it appears in different places.

Lemma 5. *If $\gamma < 1/2$, then when (1) and (3) hold, the following equations hold uniformly on $u \in [-k_n^{1/6}, k_n^{1/6}]$:*

$$V(l_n(u)) - V(l_n) = \frac{l_n V'(l_n) u[1+o(1)]}{\sqrt{k_n}}; \tag{16}$$

$$EV(l_n(u)Z) - EV(l_n Z) = \frac{u\tau_n DV(l_n Z)}{\sqrt{k_n}} + \frac{(1+\gamma) l_n V'(l_n) u^2 [1+o(1)]}{2k_n(1-\gamma)}; \tag{17}$$

$$\frac{DV(l_n Z)}{DV(l_n(u)Z)} - 1 = -\frac{u[\gamma + o(1)]}{\sqrt{k_n}}. \tag{18}$$

Proof. Since (3) holds locally uniformly, we have from (14) that for some $\theta \in [l_n, l_n(u)]$,

$$\begin{aligned} V(l_n(u)) - V(l_n) &= \int_{l_n}^{l_n(u)} V'(v)\mathrm{d}v = [l_n(u) - l_n]V'(\theta) \\ &= b_n u l_n^2(u) V'(l_n) \cdot \frac{l_n V'(\theta)}{l_n(u) V'(l_n)} \\ &= \frac{l_n V'(l_n) u[1+o(1)]}{\sqrt{k_n}}. \end{aligned}$$

This shows that (16) holds uniformly on $u \in [0, k_n^{1/6}]$. In a similar manner we can show (16) holds uniformly on $u \in [-k_n^{1/6}, 0]$.

We will only prove (17) holds uniformly on $u \in [-k_n^{1/6}, 0]$. Note that

$$\begin{aligned} EV(l_n(u)Z) - EV(l_n Z) &= \int_1^\infty \frac{V(l_n(u)v)}{v^2}\mathrm{d}v - \int_1^\infty \frac{V(l_n v)}{v^2}\mathrm{d}v \\ &= l_n(u)\int_{l_n(u)}^\infty \frac{V(s)}{s^2}\mathrm{d}s - l_n \int_{l_n}^\infty \frac{V(s)}{s^2}\mathrm{d}s \\ &= l_n(u)\int_{l_n(u)}^{l_n} \frac{V(s)}{s^2}\mathrm{d}s + b_n l_n(u) u EV(l_n Z) \\ &= d_n(u)\int_{d_n(u)}^{1} \frac{V(l_n v) - V(l_n)}{v^2}\mathrm{d}v + l_n(u) b_n u[EV(l_n Z) - V(l_n)] \\ &=: I_{n,1}(u) + I_{n,2}(u). \end{aligned}$$

From (3) and (14) we get

$$\begin{aligned} I_{n,1}(u) &= -l_n(u)V'(l_n)\int_{d_n(u)}^1 \frac{\mathrm{d}t}{t^2}\int_{d_n(u)}^1 \frac{V'(l_n s)}{V'(l_n)}\mathrm{d}s \\ &= -\frac{l_n V'(l_n) d_n(u)[1+o(1)]}{\gamma}\int_{d_n(u)}^1 \left(\frac{1}{t^{2-\gamma}} - \frac{1}{t^2}\right)\mathrm{d}t \\ &= -\frac{l_n V'(l_n) d_n(u)[1+o(1)]}{\gamma}\left\{\frac{1}{1-\gamma}\left[\frac{1}{d_n^{1-\gamma}(u)} - 1\right] - \left[\frac{1}{d_n(u)} - 1\right]\right\} \\ &= -l_n V'(l_n) d_n(u)[1+o(1)]\left\{-\frac{1}{2}\left[\frac{1}{d_n(u)} - 1\right]^2\right\} \\ &= \frac{d_n(u) l_n V'(l_n) u^2[1+o(1)]}{2k_n}; \end{aligned}$$

$$I_{n,2}(u) = \frac{\sqrt{a_n}[EV(l_n Z) - V(l_n)]u}{\sqrt{k_n}} + \frac{d_n(u) a_n u^2 [EV(l_n Z) - V(l_n)]}{k_n}.$$

Therefore we obtain from (11) that

$$EV(l_n(u)Z) - EV(l_nZ) = \frac{\sqrt{a_n}[EV(l_nZ) - V(l_n)]u}{\sqrt{k_n}} + \frac{d_n(u)l_nV'(l_n)u^2}{k_n}\left\{\frac{a_n[EV(l_nZ) - V(l_n)]}{l_nV'(l_n)} - \frac{1+o(1)}{2}\right\}$$
$$= \frac{u\tau_n DV(l_nZ)}{\sqrt{k_n}} + \frac{(1+\gamma)l_nV'(l_n)u^2[1+o(1)]}{2k_n(1-\gamma)}.$$

This shows (17).

Note that

$$\int_1^\infty t^{\gamma-2}\frac{t^\gamma - 1}{\gamma}\mathrm{d}t = \frac{1}{(1-\gamma)(1-2\gamma)}.$$

It follows from (15) with $j = 2$ that

$$\frac{T_{n,2}(u)}{[l_nV'(l_n)]^2} = \frac{2u[1+o(1)]}{\sqrt{k_n}(1-\gamma)(1-2\gamma)},$$

which together with (11), (12) and (17) yields

$$\frac{D^2V(l_n(u)Z)}{D^2V(l_nZ)} - 1 = \frac{T_n(u) - [EV(l_n(u)Z) - EV(l_nZ)]^2}{D^2V(l_nZ)} - \frac{2[EV(l_nZ) - V(l_n)][EV(l_n(u)Z) - EV(l_nZ)]}{D^2(l_nZ)}$$
$$= \frac{[l_nV'(l_n)]^2}{D^2(l_nZ)}\left\{\frac{T_n(u)}{[l_nV'(l_n)]^2} - \left[\frac{EV(l_n(u)Z) - EV(l_nZ)}{l_nV'(l_n)}\right]^2 - \frac{2[EV(l_nZ) - V(l_n)][EV(l_n(u)Z) - EV(l_nZ)]}{[l_nV'(l_n)]^2}\right\}$$
$$= \frac{2u}{\sqrt{k_n}\sigma^2(\gamma)}\left\{\frac{1+o(1)}{(1-\gamma)(1-2\gamma)} - \frac{1+o(1)}{(1-\gamma)^2} + o(1)\right\}$$
$$= \frac{u[2\gamma + o(1)]}{\sqrt{k_n}}.$$

Then from a Taylor's expansion we get

$$\frac{DV(l_nZ)}{DV(l_n(u)Z)} - 1 = \left\{1 + \frac{u[2\gamma + o(1)]}{\sqrt{k_n}}\right\}^{-1/2} - 1 = -\frac{u[\gamma + o(1)]}{\sqrt{k_n}},$$

i.e. (18) holds uniformly on $[0, k_n^{1/6}]$. In a similar way, we can show (18) holds uniformly on $[-k_n^{1/6}, 0]$. □

Lemma 6. *Suppose that (1) and (3) hold. If $\gamma < 1/2$, then there exist two functions $\alpha_{n,1}(u)$ and $\alpha_{n,2}(u)$ such that*

$$\eta_n(x,u) := \frac{DV(l_nZ)x - \sqrt{k_n}[EV(l_n(u)Z) - EV(l_nZ)]}{DV(l_n(u)Z)}$$
$$= x - \tau_n u - \alpha_{n,1}(u)xu - \alpha_{n,2}(u)u^2 \quad (19)$$

holds for all $x \in R$ and $u \in [-k_n^{1/6}, k_n^{1/6}]$, where

$$\lim_{n\to\infty}\sqrt{k_n}\alpha_{n,1}(u) = \gamma \quad \text{and} \quad \lim_{n\to\infty}\sqrt{k_n}\alpha_{n,2}(u) = [2\sigma(\gamma)]^{-1}$$

uniformly on $[-k_n^{1/6}, k_n^{1/6}]$.

Proof. Since $\tau_n = (1-2\gamma)^{1/2}[1+o(1)]$, we have from (12), (17) and (18) that

$$\eta_n(x,u) = \left\{x - \frac{\sqrt{k_n}[EV(l_n(u)Z) - EV(l_nZ)]}{DV(l_nZ)}\right\}\frac{DV(l_nZ)}{DV(l_n(u)Z)}$$
$$= \left\{x - \tau_n u - \frac{(1+\gamma)l_nV'(l_n)u^2[1+o(1)]}{2(1-\gamma)\sqrt{k_n}DV(l_nZ)}\right\}\left\{1 - \frac{u[\gamma+o(1)]}{\sqrt{k_n}}\right\}$$
$$= x - \tau_n u - \frac{(x-\tau_n u)u[\gamma + o(1)]}{\sqrt{k_n}} - \frac{(1+\gamma)u^2[1+o(1)]}{2\sqrt{k_n}(1-\gamma)\sigma(\gamma)}$$
$$= x - \tau_n u - \frac{xu[\gamma+o(1)]}{\sqrt{k_n}} - \frac{u^2[1+o(1)]}{2\sqrt{k_n}\sigma(\gamma)},$$

where all $o(1)$ terms converge to zero uniformly in u. This completes the proof of the lemma. □

For $n = 1, 2, \ldots$ and $x, u \in R$, denote

$$\hat{Z}_{n,j}(u) = \frac{V(l_n(u)Z_j) - EV(l_n(u)Z)}{\sqrt{D^2V(l_n(u)Z)}}, \quad j = 1, \ldots, k_n;$$

$$\hat{S}_{n,k_n}(u) = \frac{1}{\sqrt{k_n}} \sum_{j=1}^{k_n} \hat{Z}_{n,j}(u).$$

Proof of Theorem 1. Denote

$$\Delta_{n,1} = \sup_{x\in R} \left| \int_{|u|>k_n^{1/6}} [P(\hat{S}_{n,k_n}(u) \le \eta_n(x,u)) - \Phi(x - \tau_n u)]\phi(u)\mathrm{d}u \right|;$$

$$\Delta_{n,2} = \sup_{x\in R} \left| \int_{-k_n^{1/6}}^{k_n^{1/6}} [P(\hat{S}_{n,k_n}(u) \le \eta_n(x,u)) - \Phi(\eta_n(x,u))]\phi(u)\mathrm{d}u \right|;$$

$$\Delta_{n,3} = \sup_{x\in R} \left| \int_{-k_n^{1/6}}^{k_n^{1/6}} [\Phi(\eta_n(x,u)) - \Phi(x - \tau_n u)]\phi(u)\mathrm{d}u \right|.$$

Since

$$\Phi\left(\frac{x}{\sqrt{1+\tau_n^2}}\right) = \int_{-\infty}^{\infty} \Phi(x \pm \tau_n u)\phi(u)\mathrm{d}u, \tag{20}$$

we have from (13) and (19) that

$$\begin{aligned}\sup_{x\in R} |\Delta_n(x)| = \sup_{x\in R} \left|\Delta_n\left(\frac{x}{\sqrt{1+\tau_n^2}}\right)\right| &\le \left|\sup_{x\in R} \int_{-\infty}^{\infty} [P(\hat{S}_{n,k_n}(u) \le \eta_n(x,u)) - \Phi(x - \tau_n u)]\phi(u)\mathrm{d}u\right| + \frac{C}{\sqrt{k_n}} \\ &\le \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3} + \frac{C}{\sqrt{k_n}}.\end{aligned}$$

From the Berry–Esseen theorem (cf. [11]), we get for any $\epsilon \in (0, 1/2]$

$$\begin{aligned}\Delta_{n,2} &\le \sup_{|u|\le k_n^{1/6}} \sup_{x\in R} |P(\hat{S}_{n,k_n}(u) \le \eta_n(x,u)) - \Phi(\eta_n(x,u))| \\ &\le Ck_n^{-\epsilon} \sup_{|u|\le k_n^{1/6}} E|\hat{Z}_{n,1}(u)|^{2(1+\epsilon)} \\ &\le Ck_n^{-\epsilon}\left[\sup_{|u|\le k_n^{1/6}} E\left|\frac{V(l_n(u)Z) - V(l_n(u))}{\sqrt{D^2V(l_n(u)Z)}}\right|^{2(1+\epsilon)} + \sup_{|u|\le k_n^{1/6}}\left|\frac{EV(l_n(u)Z) - V(l_n(u))}{\sqrt{D^2V(l_n(u)Z)}}\right|^{2(1+\epsilon)}\right] \\ &=: Ck_n^{-\epsilon}(J_{n,1} + J_{n,2}).\end{aligned}$$

By (8), for any $\delta \in (0, \gamma)$, there exists a positive integer n_0 such that

$$\left|\frac{V(l_n(u)Z) - V(l_n(u))}{l_n(u)V'(l_n(u))}\right|^{2(1+\epsilon)} \le (1+\delta)^{2(1+\epsilon)} \left(\frac{Z^{\gamma+\delta} - 1}{\gamma+\delta}\right)^{2(1+\epsilon)} \tag{21}$$

holds for all $u \in [-k_n^{1/6}, k_n^{1/6}]$ and $n \ge n_0$. Since (5) holds, we can choose $\delta \in (0, \gamma)$ such that $2(1+\epsilon)(\gamma+\delta) < 1$. Then we have

$$E\left|\frac{V(l_n(u)Z) - V(l_n(u))}{l_nV'(l_n)}\right|^{2(1+\epsilon)} \le (1+\delta)^{2(1+\epsilon)} E\left(\frac{Z^{\gamma+\delta} - 1}{\gamma+\delta}\right)^{2(1+\epsilon)} < \infty. \tag{22}$$

Hence, from (12) and (17) we obtain $J_{n,1} = O(1)$. Moreover, since

$$\begin{aligned}J_{n,2} \le C\Bigg\{&\sup_{|u|\le k_n^{1/6}}\left|\frac{EV(l_n(u)Z) - EV(l_nZ)}{DV(l_n(u)Z)}\right|^{2(1+\epsilon)} + \sup_{|u|\le k_n^{1/6}}\left|\frac{EV(l_nZ) - V(l_n)}{DV(l_n(u)Z)}\right|^{2(1+\epsilon)} \\ &+ \sup_{|u|\le k_n^{1/6}}\left|\frac{V(l_n(u)) - V(l_n)}{DV(l_n(u)Z)}\right|^{2(1+\epsilon)}\Bigg\},\end{aligned}$$

we conclude from (11), (12) and (15)–(17) that $J_{n,2} = O(1)$. Hence $\Delta_{n,2} = O(k_n^{-\epsilon})$.

It is easily seen from (19) that for all $u \in [-k_n^{1/6}, k_n^{1/6}]$

$$\sup_{x\in R} |\Phi(\eta_n(x,u)) - \Phi(x - \tau_n u - \alpha_{n,1}(u)xu)| \le \frac{Cu^2}{\sqrt{k_n}}. \tag{23}$$

On the other hand, by noting that

$$\lambda\alpha_{n,1}(u)|u| \le \frac{C}{k_n^{1/3}}, \quad \text{for all } |u| \le k_n^{1/6} \text{ and } \lambda \in [0,1],$$

we get

$$\begin{aligned}
|x| &= |x[1-\lambda\alpha_{n,1}(u)u]||1-\lambda\alpha_{n,1}(u)u|^{-1} \\
&\le [|x[1-\lambda\alpha_{n,1}(u)u] - \tau_n u| + \tau_n|u|]\left|1 - \frac{C}{k_n^{1/3}}\right|^{-1} \\
&\le C[|x[1-\lambda\alpha_{n,1}(u)u] - \tau_n u| + \tau_n|u|]
\end{aligned}$$

for all $x \in R$, $u \in [-k_n^{1/6}, k_n^{1/6}]$, $\lambda \in [0,1]$ and all large n. Therefore by the mean value theorem, there exists $\lambda_n \in [0,1]$ such that

$$\begin{aligned}
&\sup_{x\in R} |\Phi(x[1-\alpha_{n,1}(u)u] - \tau_n u) - \Phi(x-\tau_n u)| = \sup_{x\in R} \alpha_{n,1}(u)|xu|\phi(x[1-\lambda_n\alpha_{n,1}(u)u] - \tau_n u) \\
&\quad \le \sup_{x\in R} \frac{C|u|}{\sqrt{k_n}}[|x[1-\lambda_n\alpha_{n,1}(u)u] - \tau_n u|\phi(x[1-\lambda_n\alpha_{n,1}(u)u] - \tau_n u) + \tau_n|u|\phi(x[1-\lambda_n\alpha_{n,1}(u)u] - \tau_n u)] \\
&\quad \le \frac{C(|u|+u^2)}{\sqrt{k_n}},
\end{aligned}$$

which coupled with (23) yields

$$\Delta_{n,3} \le \frac{C}{\sqrt{k_n}} \int_{-\infty}^{\infty} (|u|+u^2)\phi(u)\mathrm{d}u = O\left(\frac{1}{\sqrt{k_n}}\right).$$

Finally, by combining the above arguments with the fact that

$$\Delta_{n,1} \le \int_{|u|>k_n^{1/6}} \phi(u)\mathrm{d}u = O\left(\frac{1}{k_n}\right)$$

we complete the proof of the theorem. □

3. Proof of Theorem 2

For $n = 1, 2, \ldots$ and $t, x, u \in R$, let

$$\rho(\gamma) = \frac{2(1+\gamma)\sqrt{1-2\gamma}}{1-3\gamma};$$

$$G(x) = \Phi(x) - \frac{\rho(\gamma)(x^2-1)\phi(x)}{3\sqrt{k_n}};$$

$$F_n(x,u) = P(\hat{S}_{n,k_n}(u) \le x);$$

$$f_n(t,u) = E\exp\{it\hat{Z}_{n,1}(u)\} \quad \text{(the characteristic function of } \hat{Z}_{n,1}(u));$$

$$\psi_n(t,u) = \log f_n\left(\frac{t}{\sqrt{k_n}}, u\right) + \frac{t^2}{2k_n}.$$

Lemma 7. *Suppose (1) and (3) hold. Then*

$$\lim_{n\to\infty} \sup_{|u|\le k_n^{1/6}} |E\hat{Z}_{n,1}^3(u) - \rho(\gamma)| = 0, \tag{24}$$

and for any positive p with $\gamma p < 1 - 3\gamma$

$$\limsup_{n\to\infty} \sup_{|u|\le k_n^{1/6}} E|\hat{Z}_{n,1}(u)|^{3+p} < \infty. \tag{25}$$

Proof. Write

$$E[V(l_n(u)Z)-EV(l_n(u)Z)]^3 = T_{n,3}(u)-3T_{n,2}(u)[EV(l_n(u)Z)-V(l_n)]+2[EV(l_n(u)Z)-V(l_n)]^3 + E[V(l_nZ)-V(l_n)]^3-3[EV(l_n(u)Z)-V(l_n)]E[V(l_nZ)-V(l_n)]^2.$$

It follows from (11), (12) and (17) that

$$\frac{EV(l_n(u)Z)-V(l_n)}{l_nV'(l_n)} = \frac{EV(l_n(u)Z)-EV(l_nZ)}{l_nV'(l_n)}+\frac{EV(l_nZ)-V(l_n)}{l_nV'(l_n)} = \frac{1+o(1)}{1-\gamma} \tag{26}$$

holds uniformly on $u\in[-k_n^{1/6},k_n^{1/6}]$. We can conclude from (15) with $j=3$ that

$$\frac{T_{n,3}(u)}{[l_nV'(l_n)]^3}=\frac{3u[1+o(1)]}{\sqrt{k_n}}\int_1^\infty t^{\gamma-2}\left(\frac{t^\gamma-1}{\gamma}\right)^2\mathrm{d}t=o(1),$$

from (26) and (15) with $j=2$ that

$$\frac{T_{n,2}(u)[E(l_n(u)Z)-V(l_n)]}{[l_nV'(l_n)]^3}=o(1),$$

from (17) that

$$\frac{[EV(l_n(u)Z)-V(l_n)]^3}{[l_nV'(l_n)]^3}=\frac{1+o(1)}{(1-\gamma)^3},$$

from (10) with $j=3$ that

$$\frac{E[V(l_nZ)-V(l_n)]^3}{[l_nV'(l_n)]^3}=\frac{6}{(1-\gamma)(1-2\gamma)(1-3\gamma)}+o(1),$$

and from (10) and (26) with $j=2$ that

$$\frac{[EV(l_n(u)Z)-V(l_n)]E[V(l_nZ)-V(l_n)]^2}{[l_nV'(l_n)]^3}=\frac{2+o(1)}{(1-\gamma)^2(1-2\gamma)}.$$

All of the above equations hold uniformly on $[-k_n^{1/6},k_n^{1/6}]$. Thus, conclusion (24) can be easily obtained from these equations. In the same fashion we can obtain conclusion (25). □

Lemma 8. *Under* (1) *and* (3), *we have*

$$\sup_{|u|\le k_n^{1/6}}\sup_{x\in R}|F_n(x,u)-G(x)|=o\left(\frac{1}{\sqrt{k_n}}\right). \tag{27}$$

Proof. It is easy to see that the distribution of $\hat{Z}_{n,1}(u)$ depends on u only through $l_n(u)$,

$$\min_{|u|\le k_n^{1/6}} l_n(u)\ge l_n(-k_n^{1/6})\to\infty,$$

and

$$\hat{Z}_{n,1}(u)\xrightarrow{d}\frac{1}{\sigma(\gamma)}\left(\frac{Z^\gamma-1}{\gamma}-\frac{1}{1-\gamma}\right).$$

The limit is a continuous random variable and hence from Theorem 1.3 in [11] we conclude that for any $0<\delta_1<\delta_2<\infty$, there exists $\delta\in(0,1)$ such that

$$\sup_{|u|\le k_n^{1/6}}|f_n(t,u)|\le\delta,\quad \forall\delta_1\le|t|\le\delta_2 \tag{28}$$

if n is large enough.

We also need the following two equations in the proof:

$$\sup_{|u|\le k_n^{1/6}}\left|\exp\{k_n\psi_n(t,u)\}-1-\frac{\rho(\gamma)(\mathrm{i}t)^3}{6\sqrt{k_n}}\right|=(|t|^3+t^6)\exp\left(\frac{t^2}{4}\right)o\left(\frac{1}{\sqrt{k_n}}\right),\quad \forall|t|\le\delta_1\sqrt{k_n} \tag{29}$$

for any $\delta_1 > 0$ and

$$\sup_{x\in R} |F_n(x,u) - G(x)| \le \frac{1}{\pi}\int_{-T}^{T}\left|\frac{1}{t}\left[f_n^{k_n}\left(\frac{t}{\sqrt{k_n}}, u\right) - \gamma(t)\right]\right| dt + \frac{24m}{\pi T}, \tag{30}$$

where $m = \sup_{x\in R}|G'(x)|$ and

$$\gamma(t) = \exp\left(-\frac{t^2}{2}\right)\left[1 + \frac{\rho(\gamma)(it)^3}{6\sqrt{k_n}}\right].$$

(30) follows from Esseen's smoothing lemma (see, eg., Theorem 5.2 in [11]). The proof of (29) is similar to estimation of I_1 in the proof of Theorem 5.22 in [11].

By using Lemma 7 and Eqs. (27)–(29), the rest of the proof is essentially the same as that of Theorem 5.22 in [11], and the details are omitted. □

Proof of Theorem 2. From (13) and (29) we get

$$\begin{aligned}
\Delta_n\left(\frac{x}{\sqrt{1+\tau_n^2}}\right) &= \int_{-\infty}^{\infty}[F_n(\eta_n(x,u),u)\phi_n(u) - \Phi(x-\tau_n u)\phi(u)]du + o\left(\frac{1}{\sqrt{k_n}}\right)\\
&= \int_{-k_n^{1/6}}^{k_n^{1/6}} G(\eta_n(x,u))\phi_n(u)du - \int_{-k_n^{1/6}}^{k_n^{1/6}}\Phi(x-\tau_n u)\phi(u)du + o\left(\frac{1}{\sqrt{k_n}}\right)\\
&= \int_{-k_n^{1/6}}^{k_n^{1/6}}[\Phi(\eta_n(x,u)) - \Phi(x-\tau_n u)]\phi_n(u)du - \frac{\rho(\gamma)}{3\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}}[\eta_n^2(x,u)-1]\phi(\eta_n(x,u))\phi_n(u)du\\
&\quad + \int_{-k_n^{1/6}}^{k_n^{1/6}}\Phi(x-\tau_n u)[\phi_n(u)-\phi(u)]du + o\left(\frac{1}{\sqrt{k_n}}\right)\\
&=: K_{n,1}(x) + K_{n,2}(x) + K_{n,3}(x) + o\left(\frac{1}{\sqrt{k_n}}\right)
\end{aligned}$$

holds uniformly on $x \in R$.

It follows from the mean-value theorem and (19) that for any $|u| \le [-k_n^{1/6}, k_n^{1/6}]$ and $x \in R$, there exists $\lambda_n \in [0,1]$ such that

$$\begin{aligned}
\Phi(\eta_n(x,u)) - \Phi(x-\tau_n u) &= -\phi(x-\tau_n u)\{\eta_n(x,u) - [x-\tau_n u]\} + \frac{1}{2}\phi'(\theta_n(x,u))\{\eta_n(x,u) - [x-\tau_n u]\}^2\\
&=: B_{n,1}(x,u) + B_{n,2}(x,u),
\end{aligned}$$

where $\theta_n(x,u) = \lambda_n\eta_n(x,u) + (1-\lambda_n)[x-\tau_n u]$. Noting that $|x|\phi(x-\tau_n u)| \le C(1+|u|)$, we have from (19) that

$$\begin{aligned}
&\left|B_{n,1}(x,u) + \frac{1}{\sqrt{k_n}}\left\{\gamma x u + \frac{u^2}{2\sigma(\gamma)}\right\}\phi(x-\tau_n)\right|\\
&= \frac{1}{\sqrt{k_n}}\left|(\alpha_{n,1}(u)-\gamma)ux\phi(x-\tau_n u) + \left[\alpha_{n,2}(u) - \frac{1}{2\sigma(\gamma)}\right]u^2\phi(x-\tau_n u)\right|\\
&\le o\left(\frac{1}{\sqrt{k_n}}\right)(|u|+u^2)
\end{aligned}$$

holds for all $|u| \le [-k_n^{1/6}, k_n^{1/6}]$ and $x \in R$. On the other hand, noting

$$x = \left(1 - \lambda_n\alpha_{n,1}(u)u\right)^{-1}\left\{\theta_n(x,u) + (\tau_n u + \lambda_n\alpha_{n,2}(u)u^2)\right\},$$

we get

$$|x| \le C[|\theta_n(x,u)| + |u|] \tag{31}$$

and therefore

$$\begin{aligned}
|B_{n,2}(x,u)| &= |\theta_n(x,u)|\phi(\theta_n(x,u))\frac{(\sqrt{k_n}\alpha_{n,1}(u)xu + \sqrt{k_n}\alpha_{n,2}(u)u^2)^2}{2k_n}\\
&\le \frac{C\{u^2x^2|\theta_n(x,u)|\phi(\theta_n(x,u)) + u^4|\theta_n(x,u)|\phi(\theta_n(x,u))\}}{2k_n}\\
&\le \frac{C(u^2+u^4)}{k_n}
\end{aligned}$$

for all $|u| \le [-k_n^{1/6}, k_n^{1/6}]$ and $x \in R$. This gives

$$\begin{aligned}
&\left|K_{n,1}(x) - \frac{1}{\sqrt{k_n}} \int_{-k_n^{1/6}}^{k_n^{1/6}} \left\{\gamma x u + \frac{u^2}{2\sigma(\gamma)}\right\} \phi(x-\tau_n u)\phi(u)\mathrm{d}u\right| \\
&\quad \le \int_{-k_n^{1/6}}^{k_n^{1/6}} \left|B_{n,1}(x,u) - \frac{1}{\sqrt{k_n}}\left\{\gamma x u + \frac{u^2}{2\sigma(\gamma)}\right\}\phi(x-\tau_n u)\right| \phi_n(u)\mathrm{d}u \\
&\qquad + \frac{1}{3k_n}\int_{-k_n^{1/6}}^{k_n^{1/6}} \left|\gamma x u + \frac{u^2}{\sigma(\gamma)}\right| |u|^3 \phi(x-\tau_n u)\phi(u)\mathrm{d}u + \int_{-k_n^{1/6}}^{k_n^{1/6}} |B_{n,2}(x,u)|\phi(u)\mathrm{d}u \\
&\quad \le o\left(\frac{1}{\sqrt{k_n}}\right)\int_{-k_n^{1/6}}^{k_n^{1/6}} (|u|+u^2)\phi_n(u)\mathrm{d}u + \frac{C}{k_n}\int_{-k_n^{1/6}}^{k_n^{1/6}} (|u|+u^2+|u|^3+u^4+|u|^5)\phi(u)\mathrm{d}u \\
&\quad = o\left(\frac{1}{\sqrt{k_n}}\right)
\end{aligned}$$

uniformly on R.

Write

$$\begin{aligned}
&[\eta_n^2(x,u)-1]\phi(\eta_n(x,u)) - [(x-\tau_n u)^2 - 1]\phi(x-\tau_n u) \\
&\quad = [\eta_n^2(x,u)-1][\phi(\eta_n(x,u)) - \phi(x-\tau_n u)] + [\eta_n^2(x,u) - (x-\tau_n u)^2]\phi(x-\tau_n u) \\
&\quad =: L_{n,1}(x,u) + L_{n,2}(x,u).
\end{aligned}$$

Note that $|\eta_n(x,u) - 1| \le C(|x|+|u|+|xu|+u^2)$. Using the mean-value theorem and a similar estimation to (31) we have

$$|L_{n,1}(x,u)| \le \frac{C(1+u^2+u^4)}{\sqrt{k_n}}, \qquad |L_{n,2}(x,u)| \le \frac{C(1+|u|+u^2)}{\sqrt{k_n}},$$

and therefore

$$\begin{aligned}
&\left|K_{n,2}(x) + \frac{\rho(\gamma)}{3\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} [(x-\tau_n u)^2 - 1]\phi(x-\tau_n u)\phi(u)\mathrm{d}u\right| \\
&\quad \le \left|K_{n,2}(x) + \frac{\rho(\gamma)}{3\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} [(x-\tau_n u)^2 - 1]\phi(x-\tau_n u)\phi_n(u)\mathrm{d}u\right| \\
&\qquad + \left|\frac{\rho(\gamma)}{9\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} [(x-\tau_n u)^2 - 1]\phi(x-\tau_n u)u^3\phi(u)\mathrm{d}u\right| \\
&\quad \le \frac{C}{k_n}\left\{\int_{-k_n^{1/6}}^{k_n^{1/6}} (|u|+u^2+|u|^3+u^4)\phi_n(u)\mathrm{d}u + 1\right\} \\
&\quad \le o\left(\frac{1}{\sqrt{k_n}}\right)
\end{aligned}$$

holds uniformly on R. Obviously, we have

$$K_{n,3} = -\frac{1}{3\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} (u^3 - 3u)\Phi(x-\tau_n u)\phi(u)\mathrm{d}u.$$

Now we can conclude that

$$\begin{aligned}
\Delta_n\left(\frac{x}{\sqrt{1+\tau_n^2}}\right) &= \frac{1}{\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} \left\{\gamma x u + \frac{u^2}{2\sigma(\gamma)} - \frac{\rho(\gamma)[(x-\tau_n u)^2 - 1]}{3}\right\}\phi(x-\tau_n u)\phi(u)\mathrm{d}u \\
&\quad - \frac{1}{3\sqrt{k_n}}\int_{-k_n^{1/6}}^{k_n^{1/6}} (u^3-3u)\Phi(x-\tau_n u)\phi(u)\mathrm{d}u + o\left(\frac{1}{\sqrt{k_n}}\right) \\
&= \frac{1}{\sqrt{k_n}}\int_{-\infty}^{\infty} \left\{\gamma x u + \frac{u^2}{2\sigma(\gamma)} - \frac{\rho(\gamma)[(x-\tau_n u)^2 - 1]}{3}\right\}\phi(x-\tau_n u)\phi(u)\mathrm{d}u \\
&\quad - \frac{1}{3\sqrt{k_n}}\int_{-\infty}^{\infty} (u^3-3u)\Phi(x-\tau_n u)\phi(u)\mathrm{d}u + o\left(\frac{1}{\sqrt{k_n}}\right).
\end{aligned}$$

Furthermore, one can verify that for any $\tau \in R$

$$\int_{-\infty}^{\infty} \phi(x-\tau u)\phi(u)\mathrm{d}u = \frac{1}{\sqrt{1+\tau^2}}\phi\left(\frac{x}{\sqrt{1+\tau^2}}\right);$$
$$\int_{-\infty}^{\infty} u\phi(x-\tau u)\phi(u)\mathrm{d}u = \frac{\tau x}{(1+\tau^2)^{3/2}}\phi\left(\frac{x}{\sqrt{1+\tau^2}}\right);$$
$$\int_{-\infty}^{\infty} u^2\phi(x-\tau u)\phi(u)\mathrm{d}u = \frac{1+\tau^2+\tau^2x^2}{(1+\tau^2)^{5/2}}\phi\left(\frac{x}{\sqrt{1+\tau^2}}\right);$$
$$\int_{-\infty}^{\infty} u^3\phi(x-\tau u)\phi(u)\mathrm{d}u = \frac{3\tau x+3\tau^3x+\tau^3x^3}{(1+\tau^2)^{7/2}}\phi\left(\frac{x}{\sqrt{1+\tau^2}}\right)$$

and

$$\int_{-\infty}^{\infty} (u^3-3u)\Phi(x-\tau u)\phi(u)\mathrm{d}u = \frac{\tau^3}{(1+\tau^2)^{3/2}}\int_{-\infty}^{x/\sqrt{1+\tau^2}} (u^3-3u)\phi(u)\mathrm{d}u.$$

Note that $\tau_n^2 \to 1-2\gamma$. After some tedious calculations, conclusion (6) follows. The details of this are omitted here. This completes the proof of the theorem. □

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References

[1] S. Csörgö, L. Horváth, D.M. Mason, What portion of the sample makes a partial sum asymptotically stable or normal? Probab. Theory Related Fields 72 (1986) 1–16.
[2] S. Csörgö, D.M. Mason, The asymptotic distribution of sums of extreme values from a regularly varying distribution, Ann. Probab. 14 (1986) 974–983.
[3] G.S. Lo, A note on the asymptotic normality of sums of extreme values, J. Statist. Plann. Inference 22 (1989) 127–136.
[4] E. Haeusler, D.M. Mason, A law of the iterated logaritheorem for sums of extreme values from a distribution with a regularly varying upper tail, Ann. Probab. 15 (1987) 932–953.
[5] S. Csörgö, E. Haeusler, D.M. Mason, The asymptotic distribution of extreme sums, Ann. Probab. 19 (1991) 783–811.
[6] S. Cheng, L. de Haan, X. Huang, Rate of convergence of intermediate order statistics, J. Theoret. Probab. 10 (1997) 1–23.
[7] J.L. Geluk, L. de Haan, Regular Variation, Extension and Tauberian Theorems, in: CWI Tracts, vol. 40, Centre for Mathematics and Computer Science, Amsterdam, 1987.
[8] S. Cheng, Approximation to the expectation of a function of order statistics and its applications, Acta Math. Appl. Sinica (English Ser.) 13 (1997) 71–86.
[9] R.D. Reiss, Approximate Distribution of Order Statistics with Applications to Nonparametric Statistics, Springer, New York, 1989.
[10] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University, New York, 1987.
[11] V.V Petrov, Limit Theorems of Probability Theory: Sequences of Independent Random Variables, Clarendon Press, Oxford, 1995.