ALMOST SURE CONVERGENCE OF 
THE STABLE TAIL EMPIRICAL 
DEPENDENCE FUNCTION IN 
MULTIVARIATE EXTREME STATISTICS*

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Abstract

In this paper we prove the almost sure convergence of the stable tail empirical dependence function for multivariate extreme values.

Key words. Multivariate extreme value, stable tail empirical dependence function, almost sure convergence

1. Introduction and Theorems

Suppose that \( F(x, y) \) is a bivariate distribution function with two continuous marginal distribution functions, say, \( F_1 \) and \( F_2 \). Define

\[
Q_i(x) = \sup\{y : 1 - F_i(y) \geq x\}, \quad 0 \leq x \leq 1, \quad i = 1, 2.
\]

\( F \) is said to have a stable tail dependence function (STDF) \( l(x, y) \) if for \( x \geq 0 \) and \( y \geq 0 \),

\[
\lim_{t \to 0} \frac{1}{t} \tilde{F}(tx, ty) = l(x, y),
\]

where \( \tilde{F}(x, y) = 1 - F(Q_1(x), Q_2(y)) \). The concept of STDF was introduced in [6].

Suppose that \((X_i, Y_i), \ i \geq 1\) is a sequence of i.i.d. random vectors with distribution \( F(x, y) \). If there exist some sequences of constants \( a_n > 0 \), \( c_n > 0 \), \( b_n \in \mathbb{R} \) and \( d_n \in \mathbb{R} \), \( n \geq 1 \), such that the normalized maxima

\[
\left\{ \left( \frac{1}{a_n} \max_{1 \leq i \leq n} X_i - b_n \right), \frac{1}{c_n} \left( \max_{1 \leq i \leq n} Y_i - d_n \right) \right\}
\]

* This research is supported by the National Natural Science Foundation of China.
converge in distribution to a non-degenerate distribution \(G(x, y)\), then we say that \(F\) is in the domain of attraction of \(G\) and \(G\) is a bivariate extreme value distribution. Suppose that \(G_1\) and \(G_2\) are marginals of the extreme distribution \(G(x, y)\). Then \(G_1\) and \(G_2\) are two univariate extreme distributions.

It is shown in [6] that \(F\) is in the domain of attraction of \(G\) if and only if (1) \(F_1\) and \(F_2\) are in the domains of \(G_1\) and \(G_2\) respectively; (2) \(F\) has an STDF \(l(x, y)\) and

\[
G(x, y) = \exp \{ -l(-\log G_1(x), -\log G_2(y)) \}.
\]

The estimates of marginals \(G_1\) and \(G_2\) can be found in [1], [3-5] and [7]. To estimate \(G\), the work left is to estimate the stable tail dependence function \(l(x, y)\). The tail empirical dependence function of \(G\), based on \((X_i, Y_i), 1 \leq i \leq n\) and suggested by (1.1), is defined by

\[
l_n(x, y) = \frac{1}{k} \sum_{j=1}^{n} I(X_j \geq X_{n-k+1,n}, \text{ or } Y_j \geq Y_{n-k+1,n})
\]

for \(0 < x < \frac{n}{k}, \ 0 < y < \frac{n}{k}\),

and \(X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}\) and \(Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}\) are the order statistics of \(X_1, X_2, \cdots, X_n\) and \(Y_1, Y_2, \cdots, Y_n\) respectively. For simplicity, we write \(X_{n+1,n} = Y_{n+1,n} = \infty\).

Huang [6] proved that for each \(T > 0\),

\[
\sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| l_n(x, y) - l(x, y) \right|
\]

converges in probability to zero under the conditions (1.1) and (1.2). He also investigated the asymptotic distribution of \(l_n(x, y)\) under certain additional conditions.

In this paper we prove the almost sure convergence of the estimator \(l_n(x, y)\) of STDF \(l(x, y)\).

**Theorem 1.1.** Suppose that (1.1) and (1.3) hold and \(k/\log \log n \to \infty\) as \(n \to \infty\), and \(l_n(x, y)\) is defined as in (1.2). Then, for every \(T > 0\),

\[
\sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| l_n(x, y) - l(x, y) \right| \to 0 \quad \text{almost surely.} \quad (1.4)
\]

We will give the proof of Theorem 1.1 in Section 3. In Section 2 we provide two lemmas needed in the proof.

With the lemmas in Section 2 we can easily prove a multivariate analogue of Theorem 1.1. To demonstrate the multivariate result we introduce the multivariate STDF. Let \(m \geq 2\) be an integer and \(F\) be an \(m\)-dimensional distribution function with \(m\) continuous marginal distributions \(F_i, 1 \leq i \leq m\). \(F\) is said to have an STDF \(L(x_1, x_2, \cdots, x_m)\) if for every vector \((x_1, x_2, \cdots, x_m) \in R^m, x_i \geq 0, i = 1, 2, \cdots, m, \)

\[
\lim_{t \to 0} \frac{1 - F(Q_1(tx_1), Q_2(tx_2), \cdots, Q_m(tx_m))}{t} = L(x_1, x_2, \cdots, x_m), \quad (1.5)
\]

where \(Q_i(x) = \sup\{y : 1 - F_i(y) \geq x\} \) for \(0 \leq x \leq 1, i = 1, 2, \cdots, m\).
Suppose that \((X^{(1)}_j, X^{(2)}_j, \ldots, X^{(m)}_j), 1 \leq j \leq n\) are \(n\) i.i.d. \(\mathbb{R}^m\)-valued random vectors with distribution \(F\), and \(X^{(i)}_{1,n} \leq X^{(i)}_{2,n} \leq \cdots \leq X^{(i)}_{n,n}\) are the order statistics of \(X^{(i)}_1, X^{(i)}_2, \ldots, X^{(i)}_n\) for each \(i, 1 \leq i \leq m\). Let \(k = k(n), n \geq 1\) satisfy (1.3) and define

\[
L_n(x_1, x_2, \ldots, x_m) = \frac{1}{k} \sum_{j=1}^{n} I \left( \bigcup_{i=1}^{m} \{ X_j^{(i)} \geq X_{n-[nx_j]+1,n}^{(i)} \} \right)
\]

for every \(0 \leq x_i \leq \frac{n}{k}, 1 \leq i \leq m\). As an estimator of \(L(x_1, x_2, \ldots, x_m)\) we have

**Theorem 1.2.** Let (1.5) and (1.3) hold. Then, for every \(T > 0\), as \(n \to \infty\)

\[
\sup_{0 \leq x_i \leq T, 1 \leq i \leq m} |L_n(x_1, x_2, \ldots, x_m) - L(x_1, x_2, \ldots, x_m)| \to 0 \quad \text{in probability.}
\]

Additionally, if \(k/\log \log n \to \infty\) as \(n \to \infty\), then, for every \(T > 0\),

\[
\sup_{0 \leq x_i \leq T, 1 \leq i \leq m} |L_n(x_1, x_2, \ldots, x_m) - L(x_1, x_2, \ldots, x_m)| \to 0 \quad \text{almost surely.}
\]

2. Lemmas

Let \(Z_1, Z_2, \ldots\) be independent \(\mathbb{R}^d\)-valued random vectors with a common distribution function. For any Borel subset \(B\) of \(\mathbb{R}^d\), set

\[
C(B) = P(Z_1 \in B), \quad C_n(B) = \frac{1}{n} \sum_{i=1}^{n} I(Z_i \in B).
\]

Let \(A\) denote any subset of the form \(\prod_{i=1}^{d} [a_i, b_i]\) in \(\mathbb{R}^d\), \(a_i < b_i, i = 1, \ldots, d\). For \(\lambda > 0\), set

\[
\psi(\lambda) = 2\lambda^{-2}((1 + \lambda) \log(1 + \lambda) - \lambda).
\]

**Lemma 2.1.** Let \(d \geq 1\) be an integer, \(0 < C(A) \leq \frac{1}{2}\) and \(0 < \delta < 1\). Then there exists a constant \(K > 0\) depending only on \(d\) and \(\delta\) such that

\[
P \left( \sup_{\tilde{A} \subseteq A} |n^{1/2}(C_n(\tilde{A}) - C(A))| \geq \lambda \right) \leq K \exp \left\{ \frac{(1 - \delta)\lambda^2}{2C(A)} \psi \left( \frac{\lambda}{n^{1/2}C(A)} \right) \right\}, \quad \lambda > 0,
\]

where \(\tilde{A}\) denotes any half-open rectangle (of the same form as \(A\)).

**Lemma 2.2.** Assume that \(Z_1 = (U_1, \ldots, U_d)\), where \(U_i\) are uniformly distributed over \((0, 1)\), \(i = 1, \ldots, d\) and \(n_1 \leq n_2\) are two positive integers. Set \(I = \{ n : n_1 \leq n \leq n_2 \}\). \(\{k(n), n \in I\}\) is a sequence of positive integers. If \(T > 0\), \(\varepsilon > 0\) and \(\alpha > 0\) are three constants such that \(\frac{n_{\varepsilon \ell}}{n_1} \leq \frac{T \max k(n)}{\max k(n)} \leq \frac{1}{4}\), \(n_2 \leq (1 + \alpha)n_1\) and \(\varepsilon - \alpha T > 0\), then

\[
P \left( \bigcup_{n \in I} \left\{ \sup_{0 \leq x_i \leq T, i = 1, \ldots, d-1} \frac{n}{k(n)} \left| C_n \left( \prod_{i=1}^{d} [0, x_i] \right) - C \left( \prod_{i=1}^{d} [0, x_i] \right) \right| > \varepsilon \right\} \right)
\]

\[
\leq C \exp \left\{ -\beta \min \{k(n) : n \in I\} \right\}\frac{1}{1 - \exp \left\{ -\beta \min \{k(n) : n \in I\} \right\}}, \quad (2.1)
\]
where \( C \) depends only on \( d \), and \( \beta = \frac{(\varepsilon-\alpha)^2}{2(\varepsilon-\alpha)} \).

**Proof.** First we define a sequence of sets \( \{I_i, 1 \leq i \leq J\} \), where \( J \leq n_2 - n_1 + 1 \) such that \( \bigcup_{i=1}^{J} I_i \). The procedure is as follows.

Set
\[
  j(1) = \min\{k(n) : n \in I\}, \quad I_1 = \{n : k(n) < 2j(1), n \in I\}.
\]

In general, after we define \( I_1, \ldots, I_l \), if \( \bigcup_{i=1}^{l} I_i = I \), then set \( J = l \); otherwise, go on to define
\[
  j(l + 1) = \min\left\{k(n) : n \in I \setminus \bigcup_{j=1}^{l} I_j\right\}, \quad I_{l+1} = \left\{n : k(n) < 2j(l + 1), n \in I \setminus \bigcup_{j=1}^{l} I_j\right\}.
\]

The procedure stops in a finite number of steps since the set \( I \) contains only \( n_2 - n_1 + 1 \) integers. It is obvious that
\[
  j(l) = \min\{k(n) : n \in I_l\}, \quad l = 1, \ldots, J
\]
and
\[
  j(l + 1) \geq 2j(l), \quad l = 1, \ldots, J - 1. \tag{2.2}
\]

Note that
\[
  P\left( \bigcup_{n \in I_l} \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) > \varepsilon \right\} \right)
\]
\[
  \leq \sum_{l=1}^{J} P\left( \bigcup_{n \in I_l} \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) > \varepsilon \right\} \right)
\]
\[
  \leq \sum_{l=1}^{J} \left\{ P\left( \bigcup_{n \in I_l} \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) > \varepsilon \right\} \right) \right. \\
  \left. + P\left( \bigcup_{n \in I_l} \left\{ \inf_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) < -\varepsilon \right\} \right) \right\}. \tag{2.3}
\]

Set \( A_n = \prod_{i=1}^{d-1} (0, 1] \times (0, \frac{Tk(n)}{n}] \), then \( C(A_n) = \frac{Tk(n)}{n} \). For each \( n \in I_l \) we have
\[
  \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) > \varepsilon \right\}
\]
\[
  = \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \sum_{j=1}^{n_2} \left( I\left( Z_j \in \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) > \varepsilon k(n) \right) \right\}
\]
\[
  \leq \left\{ \sup_{0 \leq z_i \leq 1, i = 1, \ldots, d-1} \sum_{j=1}^{n_2} \left( I\left( Z_j \in \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right) \right\}
\]
\[
  > \varepsilon k(n) - (n_2 - n_1)C(A_n) \right\}
\]
\begin{align*}
\sum_{j=1}^{n_1} \left( I\left( Z_j \in \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > (\epsilon - \alpha T) k(n) \\
\leq \left\{ \sup_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq 2T(j(l)/n_1)} n_2^{1/2} \left( C_{n_2} \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > (\epsilon - \alpha T) \frac{j(l)}{n_2^{1/2}} \right\},
\end{align*}

i.e.,
\begin{align*}
\left\{ \sup_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq T(k(n)/n)} \frac{n}{k(n)} \left( C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > \epsilon \right\}
\leq \left\{ \sup_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq 2T(j(l)/n_1)} n_2^{1/2} \left( C_{n_2} \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > (\epsilon - \alpha T) \frac{j(l)}{n_2^{1/2}} \right\}.
\end{align*}

Similarly, one can obtain for each \( n \in I_l \) that
\begin{align*}
\left\{ \inf_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq T(k(n)/n)} \frac{n}{k(n)} \left( C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) < -\epsilon \right\}
\leq \left\{ \inf_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq 2T(j(l)/n_1)} n_1^{1/2} \left( C_{n_1} \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) < (\epsilon - \alpha T) \frac{j(l)}{n_1^{1/2}} \right\}.
\end{align*}

Setting \( B = (0, 1)^{d-1} \times (0, \frac{2T(j(l))}{n_1}) \) and applying Lemma 2.1 (choosing \( \delta = \frac{1}{2} \)) for \( n = n_2 \) and \( n = n_1 \) separately,
\begin{align*}
P \left( \bigcup_{n \in I_l} \left\{ \sup_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq T(k(n)/n)} \frac{n}{k(n)} \left( C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > \epsilon \right\} \right)
\leq P \left( \sup_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq 2T(j(l)/n_1)} n_2^{1/2} \left( C_{n_2} \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) > (\epsilon - \alpha T) \frac{j(l)}{n_2^{1/2}} \right)
\leq K \exp \left\{ - \frac{(\epsilon - \alpha T)^2 (j(l))^2}{4C(B)n_2} \frac{\psi \left( (\epsilon - \alpha T)j(l) \right)}{C(B)n_2} \right\} \tag{2.4}
\end{align*}

and
\begin{align*}
P \left( \bigcup_{n \in I_l} \left\{ \inf_{0 \leq z_i \leq 1, i=1, \ldots, d-1 \atop 0 \leq z_d \leq T(k(n)/n)} \frac{n}{k(n)} \left( C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right) < -\epsilon \right\} \right)
\leq K \exp \left\{ - \frac{(\epsilon - \alpha T)^2 (j(l))^2}{4C(B)n_1} \frac{\psi \left( (\epsilon - \alpha T)j(l) \right)}{C(B)n_1} \right\} \tag{2.5}
\end{align*}

where \( K \) depends only on \( d \).
Notice that $\psi(\lambda)$ is positive and non-decreasing for $\lambda > 0$ and $C(B) = \frac{2Tj(l)}{n}$. Immediately it follows from (2.4) and (2.5) that

$$P\left( \bigcup_{n \in I} \left\{ \sup_{0 \leq x_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon \right\} \right) \leq K \exp \left\{ - \frac{(\varepsilon - \alpha T)^2}{8(1 + \alpha)T} j(l) \psi \left( \frac{2(\varepsilon - \alpha T)}{T} \right) \right\} \leq K \exp \{-\beta j(l)\}$$

and that

$$P\left( \bigcup_{n \in I} \left\{ \inf_{0 \leq x_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right| < -\varepsilon \right\} \right) \leq K \exp\{-\beta j(l)\}.$$ 

Therefore from (2.3) and the above arguments we get

$$P\left( \bigcup_{n \in I} \left\{ \sup_{0 \leq x_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon \right\} \right) \leq 2 \sum_{l=0}^{J} K \exp\{-\beta j(l)\} \leq 2K \exp\{-\beta j(1)\} \leq \frac{2K \exp\{-\beta j(1)\}}{1 - K \exp\{-\beta j(1)\}}.$$ 

This completes the proof of (2.1) by taking $C = 2K$.

**Lemma 2.3.** Assume that $Z_1 = (U_1, \ldots, U_d)$, where $U_i$ are uniformly distributed over $(0, 1)$, $i = 1, \ldots, d$ and $k = k(n), n \geq 1$ is a sequence of integers satisfying that $\frac{k}{\log \log n} \to \infty$ and $\frac{k}{n} \to 0$ as $n \to \infty$. Then, for any $T > 0$,

$$\sup_{0 \leq x_i \leq T, 1 \leq i \leq d} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} \left( 0, \frac{kx_i}{n} \right) \right) - C\left( \prod_{i=1}^{d} \left( 0, \frac{kx_i}{n} \right) \right) \right| \to 0 \quad \text{almost surely.} \quad (2.6)$$

**Proof.** In order to prove (2.6) it suffices to show that

$$P\left( \sup_{0 \leq x_i \leq T, 1 \leq i \leq d} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon, \text{ i.o.} \right) = 0 \quad (2.7)$$

holds for every $\varepsilon > 0$.

For any given $\varepsilon > 0$, choose a positive $\alpha$ small enough such that $\varepsilon - \alpha T > 0$. Set $m_l = \lceil (1 + \frac{\alpha}{3})^l \rceil$, for $l \geq 1$, where $\lceil x \rceil$ denotes the integer part of $x$. There exists an $l_0$ so that $m_l < m_{l+1} - 1$ and $m_{l+1} - 1 \leq (1 + \alpha)m_l$ hold for $l \geq l_0$.

Note that (2.7) holds if we can prove that as $l \to \infty$,

$$P\left( \bigcup_{n=m_l}^{\infty} \left\{ \sup_{0 \leq x_i \leq 1, i = 1, \ldots, d-1} \frac{n}{k(n)} \left| C_n\left( \prod_{i=1}^{d} (0, x_i) \right) - C\left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon \right\} \right) \to 0. \quad (2.8)$$
Hence our task is to show
\[ \sum_{i=l_0}^{\infty} P \left( \bigcup_{n=m_l}^{m_{l+1}-1} \left\{ \sup_{0 \leq z \leq 1, i=1, \ldots, d-1} \frac{n}{k(n)} \left| C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon \right\} \right) < \infty \] (2.9)
which guarantees (2.8).

For each \( l \geq l_0 \), set \( p_l = \min\{k(n) : m_l \leq n \leq m_{l+1} - 1\} \). Since \( \frac{k_n}{\log \log n} \to \infty \) as \( n \to \infty \), we can easily conclude that
\[ \frac{p_l}{\log l} \to \infty \quad \text{as} \quad l \to \infty. \] (2.10)

Note that \( k(n) \to 0 \) and \( k(n) \to \infty \) as \( n \to \infty \). For a large \( l \), say \( l \geq l_1 \), making use of Lemma 2.2 and (2.10), we get
\[ P \left( \bigcup_{n=m_l}^{m_{l+1}-1} \left\{ \sup_{0 \leq z \leq 1, i=1, \ldots, d-1} \frac{n}{k(n)} \left| C_n \left( \prod_{i=1}^{d} (0, x_i) \right) - C \left( \prod_{i=1}^{d} (0, x_i) \right) \right| > \varepsilon \right\} \right) \leq \frac{C \exp\{-\beta p_l\}}{1 - K \exp\{-\beta p_l\}} \leq 2C \exp\{-\beta p_l\} \leq \frac{2C}{l^2}. \]

So (2.8) is valid by (2.9). This proves the lemma.

3. Proofs of Theorems

Proof of Theorem 1.1. Set \( U_{j_1}^{(1)} = 1 - F_1(X_j) \), \( U_{j_2}^{(2)} = 1 - F_2(X_j) \), \( j \geq 1 \). Then \( \{(U_{j_1}^{(1)}, U_{j_2}^{(2)}), j \geq 1\} \) is a sequence of i.i.d. random vectors with two marginals uniformly distributed over \((0,1)\).

Write
\[ F_n(x,y) = \frac{1}{k} \sum_{j=1}^{n} I \left( U_{j_1}^{(1)} \leq \frac{k}{n} x \text{ or } U_{j_2}^{(2)} \leq \frac{k}{n} y \right), \]
\[ F_{n1}(x) = \frac{1}{k} \sum_{j=1}^{n} I \left( U_{j_1}^{(1)} \leq \frac{k}{n} x \right), \quad F_{n2}(y) = \frac{1}{k} \sum_{j=1}^{n} I \left( U_{j_2}^{(2)} \leq \frac{k}{n} y \right). \]

Then,
\[ F_n(x,y) = F_{n1}(x) + F_{n2}(y) - \frac{1}{k} \sum_{j=1}^{n} I \left( U_{j_1}^{(1)} \leq \frac{k}{n} x, U_{j_2}^{(2)} \leq \frac{k}{n} y \right). \] (3.1)

Let \( U_{i_1}^{(1)} \leq U_{i_2}^{(2)} \leq \cdots \leq U_{i_n}^{(2)} \) be the order statistics of \( U_1^{(i)}, U_2^{(i)}, \ldots, U_n^{(i)} \), \( i = 1, 2 \). For simplicity, set \( U_{0,0}^{(i)} = 0 \). Since \( F_i \) (\( i = 1, 2 \)) are continuous, it is easily proved that
\[ l_n(x,y) = F_n \left( \frac{n}{k} U_{i_1}^{(1)}, \frac{n}{k} U_{i_2}^{(2)} \right) \quad \text{almost surely} \] (3.2)

and that
\[ P \left( X_1 \leq Q_1(x), Y_1 \leq Q_2(y) \right) = P \left( U_1^{(1)} \leq x \text{ or } U_1^{(2)} \leq y \right) \] (3.3)
for \(0 \leq x \leq 1,\ 0 \leq y \leq 1\).

From (3.1) and (3.3) we have

\[
F_n(x, y) = \frac{n}{k} F\left(\frac{k}{n} x, \frac{k}{n} y\right)
\]

\[
= F_{n1}(x) - x + F_{n2}(y) - y - \frac{n}{k} \left\{ \sum_{j=1}^{n} I \left( U_j^{(1)} \leq \frac{k}{n} x, U_j^{(2)} \leq \frac{k}{n} y \right) - P \left( U_1^{(1)} \leq x, U_1^{(2)} \leq y \right) \right\}.
\]

Thus,

\[
\sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| F_n(x, y) - \frac{n}{k} F\left(\frac{k}{n} x, \frac{k}{n} y\right) \right|
\leq \sup_{0 \leq x \leq T} \left| F_{n1}(x) - x \right| + \sup_{0 \leq y \leq T} \left| F_{n2}(y) - y \right|
\]

\[
+ \sup_{0 \leq x \leq T, 0 \leq y \leq T} \frac{n}{k} \left| \sum_{j=1}^{n} I \left( U_j^{(1)} \leq \frac{k}{n} x, U_j^{(2)} \leq \frac{k}{n} y \right) - P \left( U_1^{(1)} \leq \frac{k}{n} x, U_1^{(2)} \leq \frac{k}{n} y \right) \right|.
\]

Applying Lemma 2.3 to \(d = 1\), we have, for \(i = 1, 2\)

\[
\sup_{0 \leq x \leq T} \left| F_{ni}(x) - x \right| \to 0 \quad \text{almost surely,} \quad (3.4)
\]

and doing the same to \(d = 2\),

\[
\sup_{0 \leq x \leq T, 0 \leq y \leq T} \frac{n}{k} \left| \sum_{j=1}^{n} I \left( U_j^{(1)} \leq \frac{k}{n} x, U_j^{(2)} \leq \frac{k}{n} y \right) - P \left( U_1^{(1)} \leq \frac{k}{n} x, U_1^{(2)} \leq \frac{k}{n} y \right) \right| \to 0
\]

almost surely.

Hence

\[
\lim_{n \to \infty} \sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| F_n(x, y) - \frac{n}{k} F\left(\frac{k}{n} x, \frac{k}{n} y\right) \right| = 0 \quad \text{almost surely.} \quad (3.5)
\]

Notice that \(\tilde{F}(t, t)\) is non-decreasing in \(x\) and \(y\). According to Theorem 1 of Chapter 3 in [6], \(l(x, y)\) is monotone and continuous. Fix \(T > 0\). For each integer \(m \geq 1, 1 \leq i, j \leq m\), put \(I_{i,j} = \left[\frac{(i-1)T}{m}, \frac{iT}{m}\right] \times \left[\frac{(j-1)T}{m}, \frac{jT}{m}\right]\). Noting that for \((x, y) \in I_{i,j}\),

\[
\tilde{F}\left(\frac{t(i-1)T}{m}, \frac{t(j-1)T}{m}\right) \leq \tilde{F}(tx, ty) \leq \tilde{F}\left(\frac{tT}{m}, \frac{t'T}{m}\right),
\]

we have from (1.1) that

\[
\lim_{t \to 0} \sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| \frac{1}{t} \tilde{F}(tx, ty) - l(x, y) \right|
\]

\[
= \lim_{t \to 0} \max_{1 \leq i, j \leq m} \sup_{(x, y) \in I_{i,j}} \left| \frac{1}{t} \tilde{F}(tx, ty) - l(x, y) \right|
\]

\[
\leq \lim_{t \to 0} \max_{1 \leq i, j \leq m} \left| \frac{1}{t} \tilde{F}\left(\frac{tT}{m}, \frac{t'jT}{m}\right) - l\left(\frac{tT}{m}, \frac{jT}{m}\right) \right|
\]

\[
+ \max_{1 \leq i, j \leq m} \left( l\left(\frac{tT}{m}, \frac{jT}{m}\right) - l\left(\frac{(i-1)T}{m}, \frac{(j-1)T}{m}\right) \right)
\]

\[
= \max_{1 \leq i, j \leq m} \left( l\left(\frac{tT}{m}, \frac{jT}{m}\right) - l\left(\frac{(i-1)T}{m}, \frac{(j-1)T}{m}\right) \right),
\]
which tends to zero as \( m \to \infty \) by using the continuity of \( l(x, y) \). Therefore, we get from (3.5) that
\[
\lim_{n \to \infty} \sup_{0 \leq x \leq T, 0 \leq y \leq T} |F_n(x, y) - l(x, y)| = 0 \quad \text{almost surely.} \tag{3.6}
\]

Since \( \sup_{0 \leq x \leq T} \frac{n U^{(i)}_k}{|kx|, n} = \sup_{0 \leq x \leq T} \frac{n U^{(i)}_k}{|kx|, n} \), which are bounded almost surely from [8], by replacing \( z \) in (3.4) by \( \frac{n U^{(i)}_k}{|kx|, n} \) we have
\[
\sup_{0 \leq x \leq T} \left| \frac{|kx|}{k} - \frac{n U^{(i)}_k}{|kx|, n} \right| = \sup_{0 \leq x \leq T} \left| F_n U^{(i)}_k (|kx|, n) - \frac{n U^{(i)}_k}{|kx|, n} \right|
\]
\[
\leq \sup_{0 \leq x \leq n/k U^{(i)}_k} |F_n U^{(i)}_k (x) - x| \to 0 \quad \text{almost surely,}
\]
\[i.e.\]
\[
\sup_{0 \leq x \leq n/k U^{(i)}_k} \left| x - \frac{n U^{(i)}_k}{|kx|, n} \right| \to 0 \quad \text{almost surely.}
\]

Similarly, substituting \( \frac{n U^{(1)}_k}{|kx|, n} \) and \( \frac{n U^{(2)}_k}{|ky|, n} \) for \( x \) and \( y \) in (3.6) respectively and noticing that \( l(x, y) \) is continuous, we get immediately
\[
\sup_{0 \leq x \leq T, 0 \leq y \leq T} \left| F_n \left( \frac{n U^{(1)}_k}{|kx|, n}, \frac{n U^{(2)}_k}{|ky|, n} \right) - l(x, y) \right| \to 0 \quad \text{almost surely.}
\]

The proof of Theorem 1.1 is completed in view of (3.2).

**Proof of Theorem 1.2.** The proof of the almost sure convergence is similar to that of Theorem 1.1, and the proof of convergence in probability just follows the same procedure. The details are omitted.

**Acknowledgments.** The author is grateful to Dr. Huang Xin for putting forward the problem of this paper.

**References**


