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STAT 3611 ♦ SPRING 2005

Introduction to Probability and Statistics

Midterm Exam 2, Friday, April 8, 2005

1. (10 pts) (a). Random variable X has the probability function $f(x) = \frac{1+x^2}{8}$, x=0,1,2. Calculate Var(3X+2).

x	0	1	2
f(x)	1/8	2/8	5/8

$\mu_X = 1.5,\quad \mu_{X^2} = 2.75 \quad \sigma_X^2 = 2.75 - 1.5^2 = 0.5$

Var(3X+2)=9Var(X)=9(0.5)=**<u>4.5</u>**

(b) Two discrete random variables X and Y have the following joint probability function f(x,y):

f(x,y)	x=0	1	2
y=0	0.3	0.2	0.1
1	0.2	0.2	0

Calculate the covariance Cov(X,Y).

$\mu_X = 0.6, \quad \mu_Y = 0.4, \quad \mu_{XY} = 0.2$

$Cov(X,Y) = \mu_{XY} - \mu_X \mu_Y = 0.2 - 0.6 * 0.4 = \mathbf{-0.04}$

2. (10 pts) X and Y are two random variables. We know that E(X)=1, Var(X)=9, E(Y)=3, Var(Y)=4.

(a). If X and Y are independent, calculate E(X+XY+7);

E(X+XY+7)=E(X)+E(X)E(Y)+7=1+1(3)+7=**<u>11</u>**

(b). If X and Y are not independent, but we know Cov(X,Y)=−3, calculate $\mathrm{Var}(X-2Y)$.

$$Var(X-2Y) = Var(X) + 4Var(Y) + 2(1)(-2)Cov(X,Y) = 9 + 4 * 4 + 12 = \mathbf{37}$$

3. (20 pts) A reservation service receives requests for information according to a Poisson process with an average 60 requests per hour.
   (a). What is the probability that during a 1-minute period the reservation service receives only 1 request?

   λ=1 / minute
   $p(1,1) = e^{-1} = 0.3679$

   (b). What is the probability that the service receives 4 requests during a 5-minute period?

   $$p(4,5) = e^{-5}\frac{5^4}{4!} = 0.1755$$

   (c). How many requests are expected during a 5 hour period?

   300

   (d). If the reservation service employs **six** information operators who receive requests for information independently of one another, each according to a Poisson process with an average 10 requests per hour. What is the probability that none of the six operators receives any requests during a given 1-minute period?

   Several ways: $p(0,1/6)^6 = (e^{-1/6})^6 = e^{-1} = 0.3679$
   Or $p(0,6*1/6) = p(0,1) = e^{-1} = 0.3679$

4. (14 pts) A company manufactures concrete blocks that are used for construction purposes. Suppose that the weights of the individual concrete blocks are normally distributed with mean value μ=11.0 kg and a standard deviation σ=0.3 kg.
   a. Find the probability that a randomly selected concrete block weighs less than 10.7 kg.

   P(X<10.7)=P(Z<(10.7-11)/0.3)=P(Z<-1)=**0.1587**

b. Find the weight x which is exceeded by only 15% of the concrete blocks.
$P(X>x)=0.15$ $\quad$ $P(X<x)=0.85$
$P(Z<z)=0.85 \rightarrow z=10.4$

$X=11+0.3(1.04)=$**11.312**

5. (18 pts) There is a probability of 0.6 that an oyster produces a pear with a diameter of at least 4 mm, which is consequently of commercial value. A farmer farms 100 oysters. Let X denote the numbers of the oysters that produce pearls of commercial value.

(a) (6pts) Calculate the mean and variance of random variable X.

$$\mu = np = 60, \quad \sigma^2 = npq = 24$$

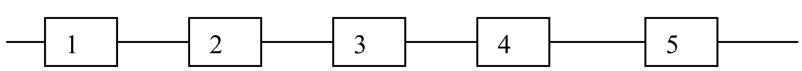
(b) (6pts)Use normal approximation to estimate the probability P(X=60).

$$P(X=60) = P(59.5 < X < 60.5) = P(\frac{59.5-60}{\sqrt{24}} < Z < \frac{60.5-60}{\sqrt{24}})$$
$$= P(= 0.1 < Z < 0.1) = 0.5398 - 0.4602 = 0.0796$$

(c) (6pts)Use normal approximation to estimate the probability P(60≤X<65).

$$P(60 \le X \le 64) = P(59.5 < X < 64.5) = P(-0.1 < Z < 0.92)$$
$$= 0.5212 - 0.4602 = 0.3610$$

6. (10 pts)A system consists of five identical components connected in series as shown:



As soon as one component fails, the entire system will fail. Suppose each component has a lifetime that is exponentially distributed with mean β=2 years and that components fail independently of one another.

a. What is the probability that the first component is still working after 1 year?

$$P(X>1) = \int_1^{\infty} \frac{1}{2} e^{-x/2} dx = e^{-0.5} = 0.6065$$

b. What is the probability that the entire system fails in 1 year?

$$1 - 0.6065^5 = 0.918$$

7. (10 pts) **a**. A couple wishes to have exactly two baby girls in their family. They will have children until this condition is fulfilled. Suppose that p=P(girl birth)=0.5. What is the probability that the family has 4 children?

   Negative binomial

$$\binom{4-1}{2-1} 0.5^2 0.5^{4-2} = 0.1875$$

   b. In a test of a particular illness, a false-positive result is obtained about 1 in 1000 times the test is administered. If the test is administered to 3000 people, fine the probability of there being more than two false-positive results.

   Poisson approximation: np=3

$$1 - \sum_{x=0}^{2} b(x, 3000, 0.001) = 1 - p(0,3) - p(1,3) - p(2,3) = 0.5768$$

8. (8 pts) A random variable X has a moment-generating function $\mathrm{M_X(t)} = 0.7e^{t^2} + 0.3e^t$.
   a. Find the mean and the variance of X by using the moment generating function.

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{M_X(t)} = 0.7e^{t^2}(2t) + 0.3e^t$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2} \mathrm{M_X(t)} = 0.7e^{t^2}(2t)^2 + 1.4e^{t^2} + 0.3e^t$$

$$\mu_1 = 0.3, \mu_2 = 1.7$$

$$\mu = 0.3, \sigma^2 = 1.7 - 0.3^2 = 1.61$$

   b. If X_1 and X_2 are two independent random variables, and both have the same distribution as that of X. Find the moment generating function for X_1+X_2+1.

$$\mathrm{M_X(t)} = 0.7e^{t^2} + 0.3e^t$$

$$\mathrm{M}_{\mathrm{X}_1+X_2+1}(\mathrm{t}) = (0.7e^{t^2} + 0.3e^t)^2 e^t$$