Example 5 **Mixture problem** A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution The interesting feature of this example is that, due to the differing rates of inflow and outflow, the volume of brine in the tank increases steadily with $V(t) = 90 + t$ gallons. The change Δx in the amount x of salt in the tank from time t to time $t + \Delta t$ (minutes) is given by

$$\Delta x \approx (4)(2)\,\Delta t - 3\left(\frac{x}{90+t}\right)\Delta t,$$

so our differential equation is

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

An integrating factor is

$$\rho(x) = \exp\left(\int \frac{3}{90+t}\,dt\right) = e^{3\ln(90+t)} = (90+t)^3,$$

which gives

$$D_t\left[(90+t)^3 x\right] = 8(90+t)^3;$$

$$(90+t)^3 x = 2(90+t)^4 + C.$$

Substitution of $x(0) = 90$ gives $C = -(90)^4$, so the amount of salt in the tank at time t is

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank is full after 30 min, and when $t = 30$, we have

$$x(30) = 2(90+30) - \frac{90^4}{120^3} \approx 202 \text{ (lb)}$$

of salt in the tank. ■

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2,\ y(0) = 0$

2. $y' - 2y = 3e^{2x},\ y(0) = 0$

3. $y' + 3y = 2xe^{-3x}$

4. $y' - 2xy = e^{x^2}$

5. $xy' + 2y = 3x,\ y(1) = 5$

6. $xy' + 5y = 7x^2,\ y(2) = 5$

7. $2xy' + y = 10\sqrt{x}$

8. $3xy' + y = 12x$

9. $xy' - y = x,\ y(1) = 7$

10. $2xy' - 3y = 9x^3$

11. $xy' + y = 3xy,\ y(1) = 0$

12. $xy' + 3y = 2x^5,\ y(2) = 1$

13. $y' + y = e^x,\ y(0) = 1$

14. $xy' - 3y = x^3,\ y(1) = 10$

15. $y' + 2xy = x,\ y(0) = -2$

16. $y' = (1 - y)\cos x,\ y(\pi) = 2$

17. $(1 + x)y' + y = \cos x,\ y(0) = 1$

18. $xy' = 2y + x^3\cos x$

19. $y' + y\cot x = \cos x$

20. $y' = 1 + x + y + xy,\ y(0) = 0$

21. $xy' = 3y + x^4\cos x,\ y(2\pi) = 0$

22. $y' = 2xy + 3x^2\exp(x^2),\ y(0) = 5$

23. $xy' + (2x - 3)y = 4x^4$

24. $(x^2 + 4)y' + 3xy = x,\ y(0) = 1$

25. $(x^2 + 1)\dfrac{dy}{dx} + 3x^3y = 6x\exp\left(-\tfrac{3}{2}x^2\right),\ y(0) = 1$

Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.

26. $(1 - 4xy^2)\dfrac{dy}{dx} = y^3$

27. $(x + ye^y)\dfrac{dy}{dx} = 1$

28. $(1 + 2xy)\dfrac{dy}{dx} = 1 + y^2$

29. Express the general solution of $dy/dx = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Fig. 1.5.9, which approaches asymptotically the graph of the equilibrium solution $x(t) \equiv 20$ that corresponds to the reservoir's long-term pollutant content. How long does it take the pollutant concentration in the reservoir to reach $10\ \mathrm{L/m^3}$?

46. The incoming water has pollutant concentration $c(t) = 10(1 + \cos t)\ \mathrm{L/m^3}$ that varies between 0 and 20, with an average concentration of $10\ \mathrm{L/m^3}$ and a period of oscillation of slightly over $6\frac{1}{4}$ months. Does it seem predictable that the lake's pollutant content should ultimately oscillate periodically about an average level of 20 million liters? Verify that the graph of $x(t)$ does, indeed, resemble the oscillatory curve shown in Fig. 1.5.9. How long does it take the pollutant concentration in the reservoir to reach $10\ \mathrm{L/m^3}$?

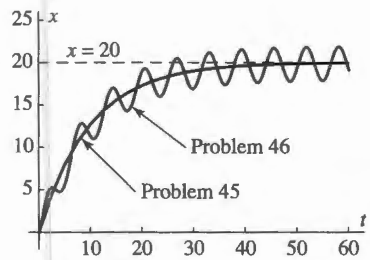

FIGURE 1.5.9. Graphs of solutions in Problems 45 and 46.

1.5 Application Indoor Temperature Oscillations

Go to goo.gl/QVuenz to download this application's computing resources including *Maple*/*Mathematica*/MATLAB/Python.

For an interesting applied problem that involves the solution of a linear differential equation, consider indoor temperature oscillations that are driven by outdoor temperature oscillations of the form

$$A(t) = a_0 + a_1 \cos \omega t + b_1 \sin \omega t. \tag{1}$$

If $\omega = \pi/12$, then these oscillations have a period of 24 hours (so that the cycle of outdoor temperatures repeats itself daily) and Eq. (1) provides a realistic model for the temperature outside a house on a day when no change in the overall day-to-day weather pattern is occurring. For instance, for a typical July day in Athens, Georgia with a minimum temperature of 70°F when $t = 4$ (4 A.M.) and a maximum of 90°F when $t = 16$ (4 P.M.), we would take

$$A(t) = 80 - 10\cos \omega(t-4) = 80 - 5\cos \omega t - 5\sqrt{3}\sin \omega t. \tag{2}$$

We derived Eq. (2) by using the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to get $a_0 = 80$, $a_1 = -5$, and $b_1 = -5\sqrt{3}$ in Eq. (1).

If we write Newton's law of cooling (Eq. (3) of Section 1.1) for the corresponding indoor temperature $u(t)$ at time t, but with the outside temperature $A(t)$ given by Eq. (1) instead of a constant ambient temperature A, we get the linear first-order differential equation

$$\frac{du}{dt} = -k(u - A(t));$$

that is,

$$\frac{du}{dt} + ku = k(a_0 + a_1 \cos \omega t + b_1 \sin \omega t) \tag{3}$$

with coefficient functions $P(t) \equiv k$ and $Q(t) = kA(t)$. Typical values of the proportionality constant k range from 0.2 to 0.5 (although k might be greater than 0.5 for a poorly insulated building with open windows, or less than 0.2 for a well-insulated building with tightly sealed windows).

1.6 Problems

Find general solutions of the differential equations in Problems 1 through 30. Primes denote derivatives with respect to x throughout.

1. $(x+y)y' = x - y$
2. $2xyy' = x^2 + 2y^2$
3. $xy' = y + 2\sqrt{xy}$
4. $(x-y)y' = x + y$
5. $x(x+y)y' = y(x-y)$
6. $(x+2y)y' = y$
7. $xy^2y' = x^3 + y^3$
8. $x^2y' = xy + x^2e^{y/x}$
9. $x^2y' = xy + y^2$
10. $xyy' = x^2 + 3y^2$
11. $(x^2 - y^2)y' = 2xy$
12. $xyy' = y^2 + x\sqrt{4x^2 + y^2}$
13. $xy' = y + \sqrt{x^2 + y^2}$
14. $yy' + x = \sqrt{x^2 + y^2}$
15. $x(x+y)y' + y(3x+y) = 0$
16. $y' = \sqrt{x+y+1}$
17. $y' = (4x+y)^2$
18. $(x+y)y' = 1$
19. $x^2y' + 2xy = 5y^3$
20. $y^2y' + 2xy^3 = 6x$
21. $y' = y + y^3$
22. $x^2y' + 2xy = 5y^4$
23. $xy' + 6y = 3xy^{4/3}$
24. $2xy' + y^3e^{-2x} = 2xy$
25. $y^2(xy' + y)(1 + x^4)^{1/2} = x$
26. $3y^2y' + y^3 = e^{-x}$
27. $3xy^2y' = 3x^4 + y^3$
28. $xe^yy' = 2(e^y + x^3e^{2x})$
29. $(2x \sin y \cos y)y' = 4x^2 + \sin^2 y$
30. $(x + e^y)y' = xe^{-y} - 1$

In Problems 31 through 42, verify that the given differential equation is exact; then solve it.

31. $(2x + 3y)\,dx + (3x + 2y)\,dy = 0$
32. $(4x - y)\,dx + (6y - x)\,dy = 0$
33. $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$
34. $(2xy^2 + 3x^2)\,dx + (2x^2y + 4y^3)\,dy = 0$
35. $\left(x^3 + \dfrac{y}{x}\right)dx + (y^2 + \ln x)\,dy = 0$
36. $(1 + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$
37. $(\cos x + \ln y)\,dx + \left(\dfrac{x}{y} + e^y\right)dy = 0$
38. $(x + \tan^{-1} y)\,dx + \dfrac{x+y}{1+y^2}\,dy = 0$
39. $(3x^2y^3 + y^4)\,dx + (3x^3y^2 + y^4 + 4xy^3)\,dy = 0$
40. $(e^x \sin y + \tan y)\,dx + (e^x \cos y + x \sec^2 y)\,dy = 0$
41. $\left(\dfrac{2x}{y} - \dfrac{3y^2}{x^4}\right)dx + \left(\dfrac{2y}{x^3} - \dfrac{x^2}{y^2} + \dfrac{1}{\sqrt{y}}\right)dy = 0$
42. $\dfrac{2x^{5/2} - 3y^{5/3}}{2x^{5/2}y^{2/3}}\,dx + \dfrac{3y^{5/3} - 2x^{5/2}}{3x^{3/2}y^{5/3}}\,dy = 0$

Find a general solution of each reducible second-order differential equation in Problems 43–54. Assume x, y and/or y′ positive where helpful (as in Example 11).

43. $xy'' = y'$
44. $yy'' + (y')^2 = 0$
45. $y'' + 4y = 0$
46. $xy'' + y' = 4x$
47. $y'' = (y')^2$
48. $x^2y'' + 3xy' = 2$
49. $yy'' + (y')^2 = yy'$
50. $y'' = (x + y')^2$
51. $y'' = 2y(y')^3$
52. $y^3y'' = 1$
53. $y'' = 2yy'$
54. $yy'' = 3(y')^2$
55. Show that the substitution $v = ax + by + c$ transforms the differential equation $dy/dx = F(ax + by + c)$ into a separable equation.
56. Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitution $v = y^{1-n}$ transforms the Bernoulli equation $dy/dx + P(x)y = Q(x)y^n$ into the linear equation
$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x).$$
57. Show that the substitution $v = \ln y$ transforms the differential equation $dy/dx + P(x)y = Q(x)(y \ln y)$ into the linear equation $dv/dx + P(x) = Q(x)v(x)$.
58. Use the idea in Problem 57 to solve the equation
$$x\frac{dy}{dx} - 4x^2y + 2y \ln y = 0.$$
59. Solve the differential equation
$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 3}$$
by finding h and k so that the substitutions $x = u + h$, $y = v + k$ transform it into the homogeneous equation
$$\frac{dv}{du} = \frac{u - v}{u + v}.$$
60. Use the method in Problem 59 to solve the differential equation
$$\frac{dy}{dx} = \frac{2y - x + 7}{4x - 3y - 18}.$$
61. Make an appropriate substitution to find a solution of the equation $dy/dx = \sin(x - y)$. Does this general solution contain the linear solution $y(x) = x - \pi/2$ that is readily verified by substitution in the differential equation?
62. Show that the solution curves of the differential equation
$$\frac{dy}{dx} = -\frac{y(2x^3 - y^3)}{x(2y^3 - x^3)}$$
are of the form $x^3 + y^3 = Cxy$.
63. The equation $dy/dx = A(x)y^2 + B(x)y + C(x)$ is called a **Riccati equation**. Suppose that one particular solution $y_1(x)$ of this equation is known. Show that the substitution
$$y = y_1 + \frac{1}{v}$$
transforms the Riccati equation into the *linear* equation
$$\frac{dv}{dx} + (B + 2Ay_1)v = -A.$$

Use the method of Problem 63 to solve the equations in Problems 64 and 65, given that $y_1(x) = x$ is a solution of each.

64. $\dfrac{dy}{dx} + y^2 = 1 + x^2$

65. $\dfrac{dy}{dx} + 2xy = 1 + x^2 + y^2$

66. An equation of the form

$$y = xy' + g(y') \tag{37}$$

is called a **Clairaut equation**. Show that the one-parameter family of straight lines described by

$$y(x) = Cx + g(C) \tag{38}$$

is a general solution of Eq. (37).

67. Consider the Clairaut equation

$$y = xy' - \tfrac{1}{4}(y')^2$$

for which $g(y') = -\frac{1}{4}(y')^2$ in Eq. (37). Show that the line

$$y = Cx - \tfrac{1}{4}C^2$$

is tangent to the parabola $y = x^2$ at the point $\left(\frac{1}{2}C, \frac{1}{4}C^2\right)$. Explain why this implies that $y = x^2$ is a singular solution of the given Clairaut equation. This singular solution and the one-parameter family of straight line solutions are illustrated in Fig. 1.6.10.

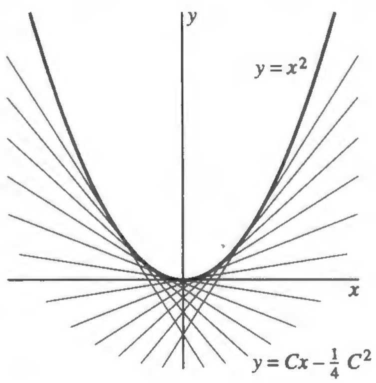

FIGURE 1.6.10. Solutions of the Clairaut equation of Problem 67. The "typical" straight line with equation $y = Cx - \frac{1}{4}C^2$ is tangent to the parabola at the point $(\frac{1}{2}C, \frac{1}{4}C^2)$.

68. Derive Eq. (18) in this section from Eqs. (16) and (17).

69. Flight trajectory In the situation of Example 7, suppose that $a = 100$ mi, $v_0 = 400$ mi/h, and $w = 40$ mi/h. Now how far northward does the wind blow the airplane?

70. Flight trajectory As in the text discussion, suppose that an airplane maintains a heading toward an airport at the origin. If $v_0 = 500$ mi/h and $w = 50$ mi/h (with the wind blowing due north), and the plane begins at the point $(200, 150)$, show that its trajectory is described by

$$y + \sqrt{x^2 + y^2} = 2(200x^9)^{1/10}.$$

71. River crossing A river 100 ft wide is flowing north at w feet per second. A dog starts at $(100, 0)$ and swims at $v_0 = 4$ ft/s, always heading toward a tree at $(0, 0)$ on the west bank directly across from the dog's starting point. **(a)** If $w = 2$ ft/s, show that the dog reaches the tree. **(b)** If $w = 4$ ft/s, show that the dog reaches instead the point on the west bank 50 ft north of the tree. **(c)** If $w = 6$ ft/s, show that the dog never reaches the west bank.

72. In the calculus of plane curves, one learns that the *curvature* κ of the curve $y = y(x)$ at the point (x, y) is given by

$$\kappa = \frac{|y''(x)|}{[1 + y'(x)^2]^{3/2}},$$

and that the curvature of a circle of radius r is $\kappa = 1/r$. [See Example 3 in Section 11.6 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition, Hoboken, NJ: Pearson, 2008.] Conversely, substitute $\rho = y'$ to derive a general solution of the second-order differential equation

$$ry'' = [1 + (y')^2]^{3/2}$$

(with r constant) in the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

Thus a circle of radius r (or a part thereof) is the *only* plane curve with constant curvature $1/r$.

Application Computer Algebra Solutions

Go to goo.gl/tLcVCl to download this application's [illegible]ng resources including *Mathematica*/MATLAB.

Computer algebra systems typically include commands for the "automatic" solution of differential equations. But two different such systems often give different results whose equivalence is not clear, and a single system may give the solution in an overly complicated form. Consequently, computer algebra solutions of differential equations often require considerable "processing" or simplification by a human user in order to yield concrete and applicable information. Here we illustrate these issues using the interesting differential equation

$$\frac{dy}{dx} = \sin(x - y) \tag{1}$$

Solution To solve the equation in (14), we separate the variables and integrate. We get

$$\int \frac{dP}{P(P-150)} = \int 0.0004\,dt,$$

$$-\frac{1}{150}\int\left(\frac{1}{P}-\frac{1}{P-150}\right)dP = \int 0.0004\,dt \qquad \text{[partial fractions]},$$

$$\ln|P| - \ln|P-150| = -0.06t + C,$$

$$\frac{P}{P-150} = \pm e^{C}e^{-0.06t} = Be^{-0.06t} \qquad [\text{where } B = \pm e^{C}]. \tag{15}$$

(a) Substitution of $t = 0$ and $P = 200$ into (15) gives $B = 4$. With this value of B we solve Eq. (15) for

$$P(t) = \frac{600e^{-0.06t}}{4e^{-0.06t}-1}. \tag{16}$$

Note that, as t increases and approaches $T = \ln(4)/0.06 \approx 23.105$, the positive denominator on the right in (16) decreases and approaches 0. Consequently $P(t) \to +\infty$ as $t \to T^{-}$. This is a *doomsday* situation—a real population *explosion*.

(b) Substitution of $t = 0$ and $P = 100$ into (15) gives $B = -2$. With this value of B we solve Eq. (15) for

$$P(t) = \frac{300e^{-0.06t}}{2e^{-0.06t}+1} = \frac{300}{2+e^{0.06t}}. \tag{17}$$

Note that, as t increases without bound, the positive denominator on the right in (16) approaches $+\infty$. Consequently, $P(t) \to 0$ as $t \to +\infty$. This is an (eventual) *extinction* situation. ■

Thus the population in Example 7 either explodes or is an endangered species threatened with extinction, depending on whether or not its initial size exceeds the threshold population $M = 150$. An approximation to this phenomenon is sometimes observed with animal populations, such as the alligator population in certain areas of the southern United States.

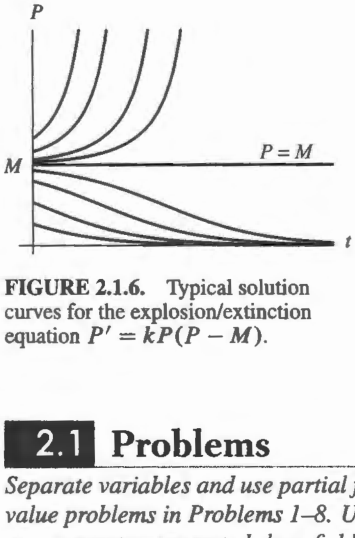

FIGURE 2.1.6. Typical solution curves for the explosion/extinction equation $P' = kP(P - M)$.

Figure 2.1.6 shows typical solution curves that illustrate the two possibilities for a population $P(t)$ satisfying Eq. (13). If $P_0 = M$ (exactly!), then the population remains constant. However, this equilibrium situation is very unstable. If P_0 exceeds M (even slightly), then $P(t)$ rapidly increases without bound, whereas if the initial (positive) population is less than M (however slightly), then it decreases (more gradually) toward zero as $t \to +\infty$. See Problem 33.

2.1 Problems

Separate variables and use partial fractions to solve the initial value problems in Problems 1–8. Use either the exact solution or a computer-generated slope field to sketch the graphs of several solutions of the given differential equation, and highlight the indicated particular solution.

1. $\dfrac{dx}{dt} = x - x^2$, $x(0) = 2$
2. $\dfrac{dx}{dt} = 10x - x^2$, $x(0) = 1$
3. $\dfrac{dx}{dt} = 1 - x^2$, $x(0) = 3$
4. $\dfrac{dx}{dt} = 9 - 4x^2$, $x(0) = 0$
5. $\dfrac{dx}{dt} = 3x(5 - x)$, $x(0) = 8$
6. $\dfrac{dx}{dt} = 3x(x - 5)$, $x(0) = 2$
7. $\dfrac{dx}{dt} = 4x(7 - x)$, $x(0) = 11$
8. $\dfrac{dx}{dt} = 7x(x - 13)$, $x(0) = 17$
9. **Population growth** The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?
10. **Extinction by disease** Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per

fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

11. **Fish population** Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$. **(a)** Show that

$$P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\right)^2,$$

where k is a constant. **(b)** If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

12. **Population growth** The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

13. **Birth rate exceeds death rate** Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$. **(a)** Show that

$$P(t) = \frac{P_0}{1 - kP_0 t}, \quad k \text{ constant.}$$

Note that $P(t) \to +\infty$ as $t \to 1/(kP_0)$. This is doomsday. **(b)** Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

14. **Death rate exceeds birth rate** Repeat part (a) of Problem 13 in the case $\beta < \delta$. What now happens to the rabbit population in the long run?

15. **Limiting population** Consider a population $P(t)$ satisfying the logistic equation $dP/dt = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = B_0 P_0 / D_0$.

16. **Limiting population** Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

17. **Limiting population** Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

18. **Threshold population** Consider a population $P(t)$ satisfying the extinction-explosion equation $dP/dt = aP^2 - bP$, where $B = aP^2$ is the time rate at which births occur and $D = bP$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$ and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the threshold population is $M = D_0 P_0 / B_0$.

19. **Threshold population** Consider an alligator population $P(t)$ satisfying the extinction-explosion equation as in Problem 18. If the initial population is 100 alligators and there are 10 births per month and 9 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10 times the threshold population M?

20. **Threshold population** Consider an alligator population $P(t)$ satisfying the extinction-explosion equation as in Problem 18. If the initial population is 110 alligators and there are 11 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10% of the threshold population M?

21. **Logistic model** Suppose that the population $P(t)$ of a country satisfies the differential equation $dP/dt = kP(200 - P)$ with k constant. Its population in 1960 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2020.

22. **Logistic model** Suppose that at time $t = 0$, half of a "logistic" population of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1000 persons per day. How long will it take for this rumor to spread to 80% of the population? (*Suggestion:* Find the value of k by substituting $P(0)$ and $P'(0)$ in the logistic equation, Eq. (3).)

23. **Solution rate** As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in a solution after t seconds satisfies the differential equation $dx/dt = 0.8x - 0.004x^2$.

    (a) What is the maximum amount of the salt that will ever dissolve in the methanol?

    (b) If $x = 50$ when $t = 0$, how long will it take for an additional 50 g of salt to dissolve?

24. **Spread of disease** Suppose that a community contains 15,000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$ the number $N(t)$ of people who have developed Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and of those who have not. How long will it take for another 5000 people to develop Michaud's syndrome?

25. **Logistic model** The data in the table in Fig. 2.1.7 are given for a certain population $P(t)$ that satisfies the logistic equation in (3). **(a)** What is the limiting population M? (*Suggestion:* Use the approximation

$$P'(t) \approx \frac{P(t+h) - P(t-h)}{2h}$$

with $h = 1$ to estimate the values of $P'(t)$ when $P = 25.00$ and when $P = 47.54$. Then substitute these values in the logistic equation and solve for k and M.) **(b)** Use the values of k and M found in part (a) to determine when $P = 75$. (*Suggestion:* Take $t = 0$ to correspond to the year 1965.)

Year	P (millions)
1964	24.63
1965	25.00
1966	25.38
⋮	⋮
2014	47.04
2015	47.54
2016	48.04

FIGURE 2.1.7. Population data for Problem 25.

26. Constant death rate A population $P(t)$ of small rodents has birth rate $\beta = (0.001)P$ (births per month per rodent) and *constant* death rate δ. If $P(0) = 100$ and $P'(0) = 8$, how long (in months) will it take this population to double to 200 rodents? (*Suggestion:* First find the value of δ.)

27. Constant death rate Consider an animal population $P(t)$ with constant death rate $\delta = 0.01$ (deaths per animal per month) and with birth rate β proportional to P. Suppose that $P(0) = 200$ and $P'(0) = 2$. **(a)** When is $P = 1000$? **(b)** When does doomsday occur?

28. Population growth Suppose that the number $x(t)$ (with t in months) of alligators in a swamp satisfies the differential equation $dx/dt = 0.0001x^2 - 0.01x$.

(a) If initially there are 25 alligators in the swamp, solve this differential equation to determine what happens to the alligator population in the long run.

(b) Repeat part (a), except with 150 alligators initially.

29. Logistic model During the period from 1790 to 1930, the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \quad P(0) = 3.9.$$

(a) What 1930 population does this logistic equation predict?

(b) What limiting population does it predict?

(c) Has this logistic equation continued since 1930 to accurately model the U.S. population?

[This problem is based on a computation by Verhulst, who in 1845 used the 1790–1840 U.S. population data to predict accurately the U.S. population through the year 1930 (long after his own death, of course).]

30. Tumor growth A tumor may be regarded as a population of multiplying cells. It is found empirically that the "birth rate" of the cells in a tumor decreases exponentially with time, so that $\beta(t) = \beta_0 e^{-\alpha t}$ (where α and β_0 are positive constants), and hence

$$\frac{dP}{dt} = \beta_0 e^{-\alpha t} P, \quad P(0) = P_0.$$

Solve this initial value problem for

$$P(t) = P_0 \exp\left(\frac{\beta_0}{\alpha}(1 - e^{-\alpha t})\right).$$

Observe that $P(t)$ approaches the finite limiting population $P_0 \exp(\beta_0/\alpha)$ as $t \to +\infty$.

31. Tumor growth For the tumor of Problem 30, suppose that at time $t = 0$ there are $P_0 = 10^6$ cells and that $P(t)$ is then increasing at the rate of 3×10^5 cells per month. After 6 months the tumor has doubled (in size and in number of cells). Solve numerically for α, and then find the limiting population of the tumor.

32. Derive the solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

of the logistic initial value problem $P' = kP(M - P)$, $P(0) = P_0$. Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.

33. (a) Derive the solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}$$

of the extinction-explosion initial value problem $P' = kP(P - M)$, $P(0) = P_0$.

(b) How does the behavior of $P(t)$ as t increases depend on whether $0 < P_0 < M$ or $P_0 > M$?

34. If $P(t)$ satisfies the logistic equation in (3), use the chain rule to show that

$$P''(t) = 2k^2 P(P - \tfrac{1}{2}M)(P - M).$$

Conclude that $P'' > 0$ if $0 < P < \frac{1}{2}M$; $P'' = 0$ if $P = \frac{1}{2}M$; $P'' < 0$ if $\frac{1}{2}M < P < M$; and $P'' > 0$ if $P > M$. In particular, it follows that any solution curve that crosses the line $P = \frac{1}{2}M$ has an inflection point where it crosses that line, and therefore resembles one of the lower S-shaped curves in Fig. 2.1.3.

35. Approach to limiting population Consider two population functions $P_1(t)$ and $P_2(t)$, both of which satisfy the logistic equation with the same limiting population M but with different values k_1 and k_2 of the constant k in Eq. (3). Assume that $k_1 < k_2$. Which population approaches M the most rapidly? You can reason *geometrically* by examining slope fields (especially if appropriate software is available), *symbolically* by analyzing the solution given in Eq. (7), or *numerically* by substituting successive values of t.

36. Logistic modeling To solve the two equations in (10) for the values of k and M, begin by solving the first equation for the quantity $x = e^{-50kM}$ and the second equation for $x^2 = e^{-100kM}$. Upon equating the two resulting expressions for x^2 in terms of M, you get an equation that is readily solved for M. With M now known, either of the original equations is readily solved for k. This technique can be used to "fit" the logistic equation to any three population values P_0, P_1, and P_2 corresponding to *equally spaced* times $t_0 = 0$, t_1, and $t_2 = 2t_1$.

37. Logistic modeling Use the method of Problem 36 to fit the logistic equation to the actual U.S. population data (Fig. 2.1.4) for the years 1850, 1900, and 1950. Solve the resulting logistic equation and compare the predicted and actual populations for the years 1990 and 2000.

38. Logistic modeling Fit the logistic equation to the actual U.S. population data (Fig. 2.1.4) for the years 1900, 1930, and 1960. Solve the resulting logistic equation, then compare the predicted and actual populations for the years 1980, 1990, and 2000.

39. Periodic growth rate Birth and death rates of animal populations typically are not constant; instead, they vary periodically with the passage of seasons. Find $P(t)$ if the population P satisfies the differential equation

$$\frac{dP}{dt} = (k + b\cos 2\pi t)P,$$

where t is in years and k and b are positive constants. Thus the growth-rate function $r(t) = k + b\cos 2\pi t$ varies periodically about its mean value k. Construct a graph that contrasts the growth of this population with one that has the same initial value P_0 but satisfies the natural growth equation $P' = kP$ (same constant k). How would the two populations compare after the passage of many years?

Application Logistic Modeling of Population Data

Go to goo.gl/s3nZZ3 to download this application's computing resources including Maple/Mathematica/MATLAB.

These investigations deal with the problem of fitting a logistic model to given population data. Thus we want to determine the numerical constants a and b so that the solution $P(t)$ of the initial value problem

$$\frac{dP}{dt} = aP + bP^2, \quad P(0) = P_0 \tag{1}$$

approximates the given values $P_0, P_1, \ldots, P_n$ of the population at the times $t_0 = 0$, $t_1, \ldots, t_n$. If we rewrite Eq. (1) (the logistic equation with $kM = a$ and $k = -b$) in the form

$$\frac{1}{P}\frac{dP}{dt} = a + bP, \tag{2}$$

then we see that the points

$$\left(P(t_i), \frac{P'(t_i)}{P(t_i)}\right), \quad i = 0, 1, 2, \ldots, n,$$

should all lie on the straight line with y-intercept a and slope b (as determined by the function of P on the right-hand side in Eq. (2)).

This observation provides a way to find a and b. If we can determine the approximate values of the derivatives $P_1', P_2', \ldots$ corresponding to the given population data, then we can proceed with the following agenda:

- First plot the points $(P_1, P_1'/P_1)$, $(P_2, P_2'/P_2)$, $\ldots$ on a sheet of graph paper with horizontal P-axis.
- Then use a ruler to draw a straight line that appears to approximate these points well.
- Finally, measure this straight line's y-intercept a and slope b.

But where are we to find the needed values of the derivative $P'(t)$ of the (as yet) unknown function P? It is easiest to use the approximation

$$P_i' = \frac{P_{i+1} - P_{i-1}}{t_{i+1} - t_{i-1}} \tag{3}$$

In Problem 27 we ask you to show that, if the projectile's initial velocity exceeds $\sqrt{2GM/R}$, then $r(t) \to \infty$ as $t \to \infty$, so it does, indeed, "escape" from the earth. With the given values of G and the earth's mass M and radius R, this gives $v_0 \approx 11{,}180$ (m/s) (about 36,680 ft/s, about 6.95 mi/s, about 25,000 mi/h).

Remark Equation (24) gives the escape velocity for any other (spherical) planetary body when we use *its* mass and radius. For instance, when we use the mass M and radius R for the moon given in Example 4, we find that escape velocity from the lunar surface is $v_0 \approx 2375$ m/s. This is just over one-fifth of the escape velocity from the earth's surface, a fact that greatly facilitates the return trip ("From the Moon to the Earth"). ■

Problems

1. The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 km/h?

Problems 2 through 8 explore the effects of resistance proportional to a power of the velocity.

2. Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$. **(a)** Show that its velocity and position at time t are given by

$$v(t) = v_0 e^{-kt}$$

and

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

(b) Conclude that the body travels only a finite distance, and find that distance.

3. Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 s later the boat has slowed to 20 ft/s. Assume, as in Problem 2, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

4. Consider a body that moves horizontally through a medium whose resistance is proportional to the *square* of the velocity v, so that $dv/dt = -kv^2$. Show that

$$v(t) = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1 + v_0 kt).$$

Note that, in contrast with the result of Problem 2, $x(t) \to +\infty$ as $t \to +\infty$. Which offers less resistance when the body is moving fairly slowly—the medium in this problem or the one in Problem 2? Does your answer seem consistent with the observed behaviors of $x(t)$ as $t \to \infty$?

5. Assuming resistance proportional to the square of the velocity (as in Problem 4), how far does the motorboat of Problem 3 coast in the first minute after its motor quits?

6. Assume that a body moving with velocity v encounters resistance of the form $dv/dt = -kv^{3/2}$. Show that

$$v(t) = \frac{4v_0}{\left(kt\sqrt{v_0} + 2\right)^2}$$

and that

$$x(t) = x_0 + \frac{2}{k}\sqrt{v_0}\left(1 - \frac{2}{kt\sqrt{v_0} + 2}\right).$$

Conclude that under a $\frac{3}{2}$-power resistance a body coasts only a finite distance before coming to a stop.

7. Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides 0.1 ft/s^2 of deceleration for each foot per second of the car's velocity. **(a)** Find the car's maximum possible (limiting) velocity. **(b)** Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

8. Rework both parts of Problem 7, with the sole difference that the deceleration due to air resistance now is $(0.001)v^2$ ft/s^2 when the car's velocity is v feet per second.

Problems 9 through 12 illustrate resistance proportional to the velocity.

9. A motorboat weighs 32,000 lb and its motor provides a thrust of 5000 lb. Assume that the water resistance is 100 pounds for each foot per second of the speed v of the boat. Then

$$1000\frac{dv}{dt} = 5000 - 100v.$$

If the boat starts from rest, what is the maximum velocity that it can attain?

10. **Falling parachutist** A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance ρv ft/s^2, taking $\rho = 0.15$ without the parachute and $\rho = 1.5$ with the parachute. (*Suggestion:* First determine her height above the ground and velocity when the parachute opens.)

11. **Falling paratrooper** According to a newspaper account, a paratrooper survived a training jump from 1200

ft when his parachute failed to open but provided some resistance by flapping unopened in the wind. Allegedly he hit the ground at 100 mi/h after falling for 8 s. Test the accuracy of this account. (*Suggestion:* Find ρ in Eq. (4) by assuming a terminal velocity of 100 mi/h. Then calculate the time required to fall 1200 ft.)

12. **Nuclear waste disposal** It is proposed to dispose of nuclear wastes—in drums with weight $W = 640$ lb and volume 8 ft^3—by dropping them into the ocean ($v_0 = 0$). The force equation for a drum falling through water is

$$m\frac{dv}{dt} = -W + B + F_R,$$

where the buoyant force B is equal to the weight (at 62.5 lb/ft^3) of the volume of water displaced by the drum (Archimedes' principle) and F_R is the force of water resistance, found empirically to be 1 lb for each foot per second of the velocity of a drum. If the drums are likely to burst upon an impact of more than 75 ft/s, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

13. Separate variables in Eq. (12) and substitute $u = v\sqrt{\rho/g}$ to obtain the upward-motion velocity function given in Eq. (13) with initial condition $v(0) = v_0$.

14. Integrate the velocity function in Eq. (13) to obtain the upward-motion position function given in Eq. (14) with initial condition $y(0) = y_0$.

15. Separate variables in Eq. (15) and substitute $u = v\sqrt{\rho/g}$ to obtain the downward-motion velocity function given in Eq. (16) with initial condition $v(0) = v_0$.

16. Integrate the velocity function in Eq. (16) to obtain the downward-motion position function given in Eq. (17) with initial condition $y(0) = y_0$.

Problems 17 and 18 apply Eqs. (12)–(17) to the motion of a crossbow bolt.

17. Consider the crossbow bolt of Example 3, shot straight upward from the ground ($y = 0$) at time $t = 0$ with initial velocity $v_0 = 49$ m/s. Take $g = 9.8$ m/s^2 and $\rho = 0.0011$ in Eq. (12). Then use Eqs. (13) and (14) to show that the bolt reaches its maximum height of about 108.47 m in about 4.61 s.

18. Continuing Problem 17, suppose that the bolt is now dropped ($v_0 = 0$) from a height of $y_0 = 108.47$ m. Then use Eqs. (16) and (17) to show that it hits the ground about 4.80 s later with an impact speed of about 43.49 m/s.

Problems 19 through 23 illustrate resistance proportional to the square of the velocity.

19. A motorboat starts from rest (initial velocity $v(0) = v_0 = 0$). Its motor provides a constant acceleration of 4 ft/s^2, but water resistance causes a deceleration of $v^2/400$ ft/s^2. Find v when $t = 10$ s, and also find the *limiting velocity* as $t \to +\infty$ (that is, the maximum possible speed of the boat).

20. An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

21. If a ball is projected upward from the ground with initial velocity v_0 and resistance proportional to v^2, deduce from Eq. (14) that the maximum height it attains is

$$y_{\max} = \frac{1}{2\rho}\ln\left(1 + \frac{\rho v_0^2}{g}\right).$$

22. Suppose that $\rho = 0.075$ (in fps units, with $g = 32$ ft/s^2) in Eq. (15) for a paratrooper falling with parachute open. If he jumps from an altitude of 10,000 ft and opens his parachute immediately, what will be his terminal speed? How long will it take him to reach the ground?

23. Suppose that the paratrooper of Problem 22 falls freely for 30 s with $\rho = 0.00075$ before opening his parachute. How long will it now take him to reach the ground?

Problems 24 through 30 explore gravitational acceleration and escape velocity.

24. The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the earth. **(a)** To what radius (in meters) would the earth have to be compressed in order for it to become a *black hole*—the escape velocity from its surface equal to the velocity $c = 3 \times 10^8$ m/s of light? **(b)** Repeat part (a) with the sun in place of the earth.

25. **(a)** Show that if a projectile is launched straight upward from the surface of the earth with initial velocity v_0 less than escape velocity $\sqrt{2GM/R}$, then the maximum distance from the center of the earth attained by the projectile is

$$r_{\max} = \frac{2GMR}{2GM - Rv_0^2},$$

where M and R are the mass and radius of the earth, respectively. **(b)** With what initial velocity v_0 must such a projectile be launched to yield a maximum altitude of 100 kilometers above the surface of the earth? **(c)** Find the maximum distance from the center of the earth, expressed in terms of earth radii, attained by a projectile launched from the surface of the earth with 90% of escape velocity.

26. Suppose that you are stranded—your rocket engine has failed—on an asteroid of diameter 3 miles, with density equal to that of the earth with radius 3960 miles. If you have enough spring in your legs to jump 4 feet straight up on earth while wearing your space suit, can you blast off from this asteroid using leg power alone?

27. **(a)** Suppose a projectile is launched vertically from the surface $r = R$ of the earth with initial velocity $v_0 = \sqrt{2GM/R}$, so $v_0^2 = k^2/R$ where $k^2 = 2GM$. Solve the differential equation $dr/dt = k/\sqrt{r}$ (from Eq. (23) in this section) explicitly to deduce that $r(t) \to \infty$ as $t \to \infty$.

(b) If the projectile is launched vertically with initial velocity $v_0 > \sqrt{2GM/R}$, deduce that

$$\frac{dr}{dt} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Why does it again follow that $r(t) \to \infty$ as $t \to \infty$?

28. **(a)** Suppose that a body is dropped ($v_0 = 0$) from a distance $r_0 > R$ from the earth's center, so its acceleration is $dv/dt = -GM/r^2$. Ignoring air resistance, show that it reaches the height $r < r_0$ at time

$$t = \sqrt{\frac{r_0}{2GM}}\left(\sqrt{rr_0 - r^2} + r_0 \cos^{-1}\sqrt{\frac{r}{r_0}}\right).$$

(*Suggestion:* Substitute $r = r_0 \cos^2\theta$ to evaluate $\int \sqrt{r/(r_0 - r)}\, dr$.) **(b)** If a body is dropped from a height of 1000 km above the earth's surface and air resistance is neglected, how long does it take to fall and with what speed will it strike the earth's surface?

29. Suppose that a projectile is fired straight upward from the surface of the earth with initial velocity $v_0 < \sqrt{2GM/R}$. Then its height $y(t)$ above the surface satisfies the initial value problem

$$\frac{d^2y}{dt^2} = -\frac{GM}{(y+R)^2}; \quad y(0) = 0, \quad y'(0) = v_0.$$

Substitute $dv/dt = v(dv/dy)$ and then integrate to obtain

$$v^2 = v_0^2 - \frac{2GMy}{R(R+y)}$$

for the velocity v of the projectile at height y. What maximum altitude does it reach if its initial velocity is 1 km/s?

30. In Jules Verne's original problem, the projectile launched from the surface of the earth is attracted by both the earth and the moon, so its distance $r(t)$ from the center of the earth satisfies the initial value problem

$$\frac{d^2r}{dt^2} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}; \quad r(0) = R, \quad r'(0) = v_0$$

where M_e and M_m denote the masses of the earth and the moon, respectively; R is the radius of the earth and $S = 384{,}400$ km is the distance between the centers of the earth and the moon. To reach the moon, the projectile must only just pass the point between the moon and earth where its net acceleration vanishes. Thereafter it is "under the control" of the moon, and falls from there to the lunar surface. Find the *minimal* launch velocity v_0 that suffices for the projectile to make it "From the Earth to the Moon."

Application Rocket Propulsion

Suppose that the rocket of Fig. 2.3.5 blasts off straight upward from the surface of the earth at time $t = 0$. We want to calculate its height y and velocity $v = dy/dt$ at time t. The rocket is propelled by exhaust gases that exit (rearward) with constant speed c (relative to the rocket). Because of the combustion of its fuel, the mass $m = m(t)$ of the rocket is variable.

To derive the equation of motion of the rocket, we use Newton's second law in the form

$$\frac{dP}{dt} = F, \tag{1}$$

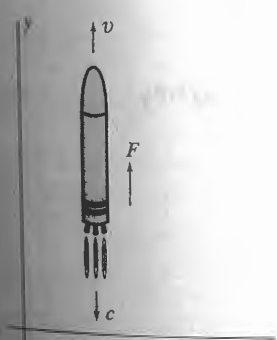

FIGURE 2.3.5. An ascending rocket.

where P is momentum (the product of mass and velocity) and F denotes net external force (gravity, air resistance, etc.). If the mass m of the rocket is constant so $m'(t) \equiv 0$—when its rockets are turned off or burned out, for instance—then Eq. (1) gives

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = m\frac{dv}{dt},$$

which (with $dv/dt = a$) is the more familiar form $F = ma$ of Newton's second law.

But here m is not constant. Suppose m changes to $m + \Delta m$ and v to $v + \Delta v$ during the short time interval from t to $t + \Delta t$. Then the change in the momentum *of the rocket itself* is

$$\Delta P \approx (m + \Delta m)(v + \Delta v) - mv = m\,\Delta v + v\,\Delta m + \Delta m\,\Delta v.$$

But the system also includes the exhaust gases expelled during this time interval, with mass $-\Delta m$ and approximate velocity $v - c$. Hence the total change in momen-

Problems

In each of Problems 1–22, use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t (as in Examples 5 and 7).

1. $\begin{aligned} x + 3y &= 9 \\ 2x + y &= 8 \end{aligned}$

2. $\begin{aligned} 3x + 2y &= 9 \\ x - y &= 8 \end{aligned}$

3. $\begin{aligned} 2x + 3y &= 1 \\ 3x + 5y &= 3 \end{aligned}$

4. $\begin{aligned} 5x - 6y &= 1 \\ 6x - 5y &= 10 \end{aligned}$

5. $\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 9 \end{aligned}$

6. $\begin{aligned} 4x - 2y &= 4 \\ 6x - 3y &= 7 \end{aligned}$

7. $\begin{aligned} x - 4y &= -10 \\ -2x + 8y &= 20 \end{aligned}$

8. $\begin{aligned} 3x - 6y &= 12 \\ 2x - 4y &= 8 \end{aligned}$

9. $\begin{aligned} x + 5y + z &= 2 \\ 2x + y - 2z &= 1 \\ x + 7y + 2z &= 3 \end{aligned}$

10. $\begin{aligned} x + 3y + 2z &= 2 \\ 2x + 7y + 7z &= -1 \\ 2x + 5y + 2z &= 7 \end{aligned}$

11. $\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$

12. $\begin{aligned} 3x + 5y - z &= 13 \\ 2x + 7y + z &= 28 \\ x + 7y + 2z &= 32 \end{aligned}$

13. $\begin{aligned} 3x + 9y + 7z &= 0 \\ 2x + 7y + 4z &= 0 \\ 2x + 6y + 5z &= 0 \end{aligned}$

14. $\begin{aligned} 4x + 9y + 12z &= -1 \\ 3x + y + 16z &= -46 \\ 2x + 7y + 3z &= 19 \end{aligned}$

15. $\begin{aligned} x + 3y + 2z &= 5 \\ x - y + 3z &= 3 \\ 3x + y + 8z &= 10 \end{aligned}$

16. $\begin{aligned} x - 3y + 2z &= 6 \\ x + 4y - z &= 4 \\ 5x + 6y + z &= 20 \end{aligned}$

17. $\begin{aligned} 2x - y + 4z &= 7 \\ 3x + 2y - 2z &= 3 \\ 5x + y + 2z &= 15 \end{aligned}$

18. $\begin{aligned} x + 5y + 6z &= 3 \\ 5x + 2y - 10z &= 1 \\ 8x + 17y + 8z &= 5 \end{aligned}$

19. $\begin{aligned} x - 2y + z &= 2 \\ 2x - y - 4z &= 13 \\ x - y - z &= 5 \end{aligned}$

20. $\begin{aligned} 2x + 3y + 7z &= 15 \\ x + 4y + z &= 20 \\ x + 2y + 3z &= 10 \end{aligned}$

21. $\begin{aligned} x + y - z &= 5 \\ 3x + y + 3z &= 11 \\ 4x + y + 5z &= 14 \end{aligned}$

22. $\begin{aligned} 4x - 2y + 6z &= 0 \\ x - y - z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}$

In each of Problems 23–30, a second-order differential equation and its general solution y(x) are given. Determine the constants A and B so as to find a solution of the differential equation that satisfies the given initial conditions involving y(0) and y′(0).

23. $y'' + 4y = 0,\ y(x) = A\cos 2x + B\sin 2x,$
$y(0) = 3,\ y'(0) = 8$

24. $y'' - 9y = 0,\ y(x) = A\cosh 3x + B\sinh 3x,$
$y(0) = 5,\ y'(0) = 12$

25. $y'' - 25y = 0,\ y(x) = Ae^{5x} + Be^{-5x},$
$y(0) = 10,\ y'(0) = 20$

26. $y'' - 121y = 0,\ y(x) = Ae^{11x} + Be^{-11x},$
$y(0) = 44,\ y'(0) = 22$

27. $y'' + 2y' - 15y = 0,\ y(x) = Ae^{3x} + Be^{-5x},$
$y(0) = 40,\ y'(0) = -16$

28. $y'' - 10y' + 21y = 0,\ y(x) = Ae^{3x} + Be^{7x},$
$y(0) = 15,\ y'(0) = 13$

29. $6y'' - 5y' + y = 0,\ y(x) = Ae^{x/2} + Be^{x/3},$
$y(0) = 7,\ y'(0) = 11$

30. $15y'' + y' - 28y = 0,\ y(x) = Ae^{4x/3} + Be^{-7x/5},$
$y(0) = 41,\ y'(0) = 164$

31. A system of the form

$$\begin{aligned} a_1x + b_1y &= 0 \\ a_2x + b_2y &= 0, \end{aligned}$$

in which the constants on the right-hand side are all zero, is said to be **homogeneous**. Explain by geometric reasoning why such a system has either a unique solution or infinitely many solutions. In the former case, what is the unique solution?

32. Consider the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

of two equations in three unknowns.

(a) Use the fact that the graph of each such equation is a plane in *xyz*-space to explain why such a system always has either no solution or infinitely many solutions.

(b) Explain why the system must have infinitely many solutions if $d_1 = 0 = d_2$.

33. The linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

of three equations in two unknowns represents three lines L_1, L_2, and L_3 in the *xy*-plane. Figure 3.1.5 shows six possible configurations of these three lines. In each case describe the solution set of the system.

34. Consider the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

of three equations in three unknowns to represent three planes P_1, P_2, and P_3 in *xyz*-space. Describe the solution set of the system in each of the following cases.

(a) The three planes are parallel and distinct.

(b) The three planes coincide—$P_1 = P_2 = P_3$.

(c) P_1 and P_2 coincide and are parallel to P_3.

(d) P_1 and P_2 intersect in a line L that is parallel to P_3.

(e) P_1 and P_2 intersect in a line L that lies in P_3.

(f) P_1 and P_2 intersect in a line L that intersects P_3 in a single point.

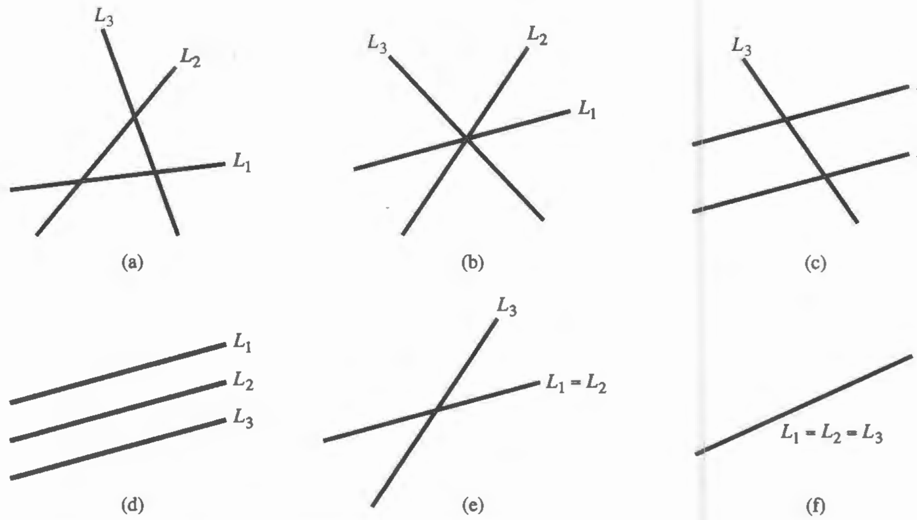

FIGURE 3.1.5. Three lines in the plane (Problem 33).

3.2 Matrices and Gaussian Elimination

In Example 6 of Section 3.1 we applied the method of elimination to solve the linear system

$$\begin{aligned} \mathbf{1}x + \mathbf{2}y + \mathbf{1}z &= \mathbf{4} \\ \mathbf{3}x + \mathbf{8}y + \mathbf{7}z &= \mathbf{20} \\ \mathbf{2}x + \mathbf{7}y + \mathbf{9}z &= \mathbf{23}. \end{aligned} \tag{1}$$

There we employed elementary operations to transform this system into the equivalent system

$$\begin{aligned} \mathbf{1}x + \mathbf{2}y + \mathbf{1}z &= \mathbf{4} \\ \mathbf{0}x + \mathbf{1}y + \mathbf{2}z &= \mathbf{4} \\ \mathbf{0}x + \mathbf{0}y + \mathbf{1}z &= \mathbf{3}, \end{aligned} \tag{2}$$

which we found easy to solve by back substitution. Here we have printed in color the coefficients and constants (including the 0s and 1s that would normally be omitted) because everything else—the symbols x, y, and z for the variables and the $+$ and $=$ signs—is excess baggage that means only extra writing, for we can keep track of these symbols mentally. In effect, in Example 6 we used an appropriate sequence of operations to transform the array

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \tag{3}$$

of coefficients and constants in (1) into the array

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \tag{4}$$

of constants and coefficients in (2).

as follows:

$$\begin{aligned} x_1 &= 5 + 2s - 3t \\ x_2 &= s \\ x_3 &= -3 - 2t \\ x_4 &= 7 + 4t \\ x_5 &= t. \end{aligned} \tag{19}$$

Thus, the substitution of any two specific values for s and t in (19) yields a particular solution $(x_1, x_2, x_3, x_4, x_5)$ of the system, and each of the system's infinitely many different solutions is the result of some such substitution. ■

Examples 3 and 5 illustrate the ways in which Gaussian elimination can result in either a unique solution or infinitely many solutions. On the other hand, if the reduction of the augmented matrix to echelon form leads to a row of the form

$$0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad *,$$

where the asterisk denotes a nonzero entry in the last column, then we have an inconsistent equation,

$$0x_1 + 0x_2 + \cdots + 0x_n = *,$$

and therefore the system has no solution.

Remark We use algorithms such as the back substitution and Gaussian elimination algorithms of this section to outline the basic computational procedures of linear algebra. In modern numerical work, these procedures often are implemented on a computer. For instance, linear systems of more than four equations are usually solved in practice by using a computer to carry out the process of Gaussian elimination. ■

3.2 Problems

The linear systems in Problems 1–10 are in echelon form. Solve each by back substitution.

1. $\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_2 + 3x_3 &= 6 \\ x_3 &= 2 \end{aligned}$

2. $\begin{aligned} 2x_1 - 5x_2 + x_3 &= 2 \\ 3x_2 - 2x_3 &= 9 \\ x_3 &= -3 \end{aligned}$

3. $\begin{aligned} x_1 - 3x_2 + 4x_3 &= 7 \\ x_2 - 5x_3 &= 2 \end{aligned}$

4. $\begin{aligned} x_1 - 5x_2 + 2x_3 &= 10 \\ x_2 - 7x_3 &= 5 \end{aligned}$

5. $\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= 9 \\ x_2 - x_3 + 2x_4 &= 1 \\ x_3 - 3x_4 &= 5 \end{aligned}$

6. $\begin{aligned} x_1 - 2x_2 + 5x_3 - 3x_4 &= 7 \\ x_2 - 3x_3 + 2x_4 &= 3 \\ x_4 &= -4 \end{aligned}$

7. $\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_2 - 2x_3 + 7x_4 &= 7 \end{aligned}$

8. $\begin{aligned} x_1 - 10x_2 + 3x_3 - 13x_4 &= 5 \\ x_3 + 3x_4 &= 10 \end{aligned}$

9. $\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 6 \\ 3x_2 - x_3 - 2x_4 &= 2 \\ 3x_3 + 4x_4 &= 9 \\ x_4 &= 6 \end{aligned}$

10. $\begin{aligned} x_1 - 5x_2 + 2x_3 - 7x_4 + 11x_5 &= 0 \\ x_2 - 13x_3 + 3x_4 - 7x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$

In Problems 11–22, use elementary row operations to transform each augmented coefficient matrix to echelon form. Then solve the system by back substitution.

11. $\begin{aligned} 2x_1 + 8x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_1 + 7x_2 + 4x_3 &= 8 \end{aligned}$

12. $\begin{aligned} 3x_1 + x_2 - 3x_3 &= 6 \\ 2x_1 + 7x_2 + x_3 &= -9 \\ 2x_1 + 5x_2 &= -5 \end{aligned}$

13. $\begin{aligned} x_1 + 3x_2 + 3x_3 &= 13 \\ 2x_1 + 5x_2 + 4x_3 &= 23 \\ 2x_1 + 7x_2 + 8x_3 &= 29 \end{aligned}$

14. $\begin{aligned} 3x_1 - 6x_2 - 2x_3 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 17 \\ x_1 - 2x_2 - 2x_3 &= -9 \end{aligned}$

15. $\begin{aligned} 3x_1 + x_2 - 3x_3 &= -4 \\ x_1 + x_2 + x_3 &= 1 \\ 5x_1 + 6x_2 + 8x_3 &= 8 \end{aligned}$

16. $\begin{aligned} 2x_1 + 5x_2 + 12x_3 &= 6 \\ 3x_1 + x_2 + 5x_3 &= 12 \\ 5x_1 + 8x_2 + 21x_3 &= 17 \end{aligned}$

17. $\begin{aligned} x_1 - 4x_2 - 3x_3 - 3x_4 &= 4 \\ 2x_1 - 6x_2 - 5x_3 - 5x_4 &= 5 \\ 3x_1 - x_2 - 4x_3 - 5x_4 &= -7 \end{aligned}$

18. $\begin{aligned} 3x_1 - 6x_2 + x_3 + 13x_4 &= 15 \\ 3x_1 - 6x_2 + 3x_3 + 21x_4 &= 21 \\ 2x_1 - 4x_2 + 5x_3 + 26x_4 &= 23 \end{aligned}$

19. $$\begin{aligned} 3x_1 + x_2 + x_3 + 6x_4 &= 14 \\ x_1 - 2x_2 + 5x_3 - 5x_4 &= -7 \\ 4x_1 + x_2 + 2x_3 + 7x_4 &= 17 \end{aligned}$$

20. $$\begin{aligned} 2x_1 + 4x_2 - x_3 - 2x_4 + 2x_5 &= 6 \\ x_1 + 3x_2 + 2x_3 - 7x_4 + 3x_5 &= 9 \\ 5x_1 + 8x_2 - 7x_3 + 6x_4 + x_5 &= 4 \end{aligned}$$

21. $$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 6 \\ 2x_1 - 2x_2 - 5x_3 \qquad &= -13 \\ 3x_1 \qquad + x_3 + x_4 &= 13 \\ 4x_1 - 2x_2 - 3x_3 + x_4 &= 1 \end{aligned}$$

22. $$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 - 5x_3 \qquad &= -10 \\ 4x_1 + x_2 + 2x_3 + x_4 &= 17 \\ 3x_1 \qquad + x_3 + x_4 &= 12 \end{aligned}$$

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

23. $$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= k \end{aligned}$$

24. $$\begin{aligned} 3x + 2y &= 0 \\ 6x + ky &= 0 \end{aligned}$$

25. $$\begin{aligned} 3x + 2y &= 11 \\ 6x + ky &= 21 \end{aligned}$$

26. $$\begin{aligned} 3x + 2y &= 1 \\ 7x + 5y &= k \end{aligned}$$

27. $$\begin{aligned} x + 2y + z &= 3 \\ 2x - y - 3z &= 5 \\ 4x + 3y - z &= k \end{aligned}$$

28. Under what condition on the constants a, b, and c does the system
$$\begin{aligned} 2x - y + 3z &= a \\ x + 2y + z &= b \\ 7x + 4y + 9z &= c \end{aligned}$$
have a unique solution? No solution? Infinitely many solutions?

29. This problem deals with the reversibility of elementary row operations.
   (a) If the elementary row operation cR_p changes the matrix **A** to the matrix **B**, show that $(1/c)R_p$ changes **B** to **A**.
   (b) If SWAP(R_p, R_q) changes **A** to **B**, show that SWAP(R_p, R_q) also changes **B** to **A**.
   (c) If $cR_p + R_q$ changes **A** to **B**, show that $(-c)R_p + R_q$ changes **B** to **A**.
   (d) Conclude that if **A** can be transformed into **B** by a finite sequence of elementary row operations, then **B** can similarly be transformed into **A**.

30. This problem outlines a proof that two linear systems LS_1 and LS_2 are equivalent (that is, have the same solution set) if their augmented coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are row equivalent.
   (a) If a *single* elementary row operation transforms $\mathbf{A}_1$ to $\mathbf{A}_2$, show directly—considering separately the three cases—that every solution of LS_1 is also a solution of LS_2.
   (b) Explain why it now follows from Problem 29 that every solution of either system is also a solution of the other system; thus the two systems have the same solution set.

Application Automated Row Reduction

Go to goo.gl/9szcxW to download this application's computing resources including Mathematica/MATLAB.

Computer algebra systems are often used to ease the labor of matrix computations, including elementary row operations. The 3×4 augmented coefficient matrix of Example 3 can be entered with the *Maple* command

```
with(linalg):
A := array( [[1, 2, 1,  4],
             [3, 8, 7, 20],
             [2, 7, 9, 23]] );
```

or the *Mathematica* command

```
A = {{1, 2, 1,  4},
     {3, 8, 7, 20},
     {2, 7, 9, 23}}
```

or the MATLAB command

```
A = [1 2 1  4
     3 8 7 20
     2 7 9 23]
```

The *Maple* `linalg` package has built-in elementary row operations that can be used to carry out the reduction of `A` exhibited in Example 3, as follows:

Such a (square) matrix, with ones on its **principal diagonal** (the one from upper left to lower right) and zeros elsewhere, is called an **identity matrix** (for reasons given in Section 3.4). For instance, the 2×2 and 3×3 identity matrices are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix in (9) is the $n \times n$ identity matrix. With this terminology, the preceding argument establishes the following theorem.

THEOREM 4 Homogeneous Systems with Unique Solutions

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the homogeneous system with coefficient matrix $\mathbf{A}$ has only the trivial solution if and only if $\mathbf{A}$ is row equivalent to the $n \times n$ identity matrix.

Example 5 The computation in Example 2 (disregarding the fourth column in each matrix there) shows that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}$$

is row equivalent to the 3×3 identity matrix. Hence Theorem 4 implies that the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 3x_1 + 8x_2 + 7x_3 &= 0 \\ 2x_1 + 7x_2 + 9x_3 &= 0 \end{aligned}$$

with coefficient matrix $\mathbf{A}$ has only the trivial solution $x_1 = x_2 = x_3 = 0$. ■

3.3 Problems

Find the reduced echelon form of each of the matrices given in Problems 1–20.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 7 & 15 \\ 2 & 5 & 11 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 7 & -1 \\ 5 & 2 & 8 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -11 \\ 2 & 3 & -19 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 & 19 \\ 4 & -7 & 70 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 2 & -5 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -4 & -2 \\ 3 & -12 & 1 \\ 2 & -8 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 7 & 4 & 0 \\ 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 5 & 2 & 3 \\ 2 & 7 & 7 & 22 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 2 & 4 & 2 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & -3 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 1 & -1 & -4 \\ 1 & -2 & -2 & 8 & -1 \\ 2 & 3 & -1 & 3 & 11 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 2 & 3 & 18 & 11 & 9 \\ 2 & 5 & 26 & 21 & 11 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 6 & 1 & 7 & 13 \\ 5 & 10 & 8 & 18 & 47 \\ 2 & 4 & 5 & 9 & 26 \end{bmatrix}$

21–30. Use the method of Gauss-Jordan elimination (transforming the augmented matrix into reduced echelon form) to solve Problems 11–20 in Section 3.2.

31. Show that the two matrices in (1) are both row equivalent to the 3×3 identity matrix (and hence, by Theorem 1, to each other).

32. Show that the 2×2 matrix
$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
is row equivalent to the 2×2 identity matrix provided that $ad - bc \neq 0$.

33. List all possible reduced row-echelon forms of a 2×2 matrix, using asterisks to indicate elements that may be either zero or nonzero.

34. List all possible reduced row-echelon forms of a 3×3 matrix, using asterisks to indicate elements that may be either zero or nonzero.

35. Consider the homogeneous system
$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$
(a) If $x = x_0$ and $y = y_0$ is a solution and k is a real number, then show that $x = kx_0$ and $y = ky_0$ is also a solution.
(b) If $x = x_1$, $y = y_1$ and $x = x_2$, $y = y_2$ are both solutions, then show that $x = x_1 + x_2$, $y = y_1 + y_2$ is a solution.

36. Suppose that $ad - bc \neq 0$ in the homogeneous system of Problem 35. Use Problem 32 to show that its only solution is the trivial solution.

37. Show that the homogeneous system in Problem 35 has a nontrivial solution if and only if $ad - bc = 0$.

38. Use the result of Problem 37 to find all values of c for which the homogeneous system
$$\begin{aligned} (c+2)x + \quad 3y &= 0 \\ 2x + (c-3)y &= 0 \end{aligned}$$
has a nontrivial solution.

39. Consider a *homogeneous* system of three equations in three unknowns. Suppose that the third equation is the sum of some multiple of the first equation and some multiple of the second equation. Show that the system has a nontrivial solution.

40. Let $\mathbf{E}$ be an echelon matrix that is row equivalent to the matrix $\mathbf{A}$. Show that $\mathbf{E}$ has the same number of nonzero rows as does the reduced echelon form $\mathbf{E}^*$ of $\mathbf{A}$. Thus the number of nonzero rows in an echelon form of $\mathbf{A}$ is an "invariant" of the matrix $\mathbf{A}$. *Suggestion*: Consider reducing $\mathbf{E}$ to $\mathbf{E}^*$.

Application Automated Row Reduction

Go to goo.gl/7ryusP to download this application's ...ng resources including ...athematica/MATLAB.

Most computer algebra systems include commands for the immediate reduction of matrices to reduced echelon form. For instance, if the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$
of Example 2 has been entered—as illustrated in the 3.2 Application—then the *Maple* command

```
with(linalg):  R := rref(A);
```

or the *Mathematica* command

```
R = RowReduce[A] // MatrixForm
```

or the MATLAB command

```
R = rref(A)
```

or the Wolfram|Alpha query

```
row reduce ((1, 2, 1, 4), (3, 8, 7, 20), (2, 7, 9, 23))
```

produces the reduced echelon matrix
$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

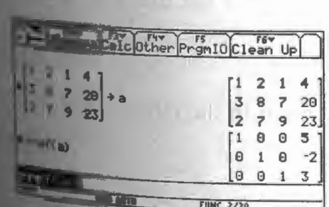

FIGURE 3.3.1. Finding a reduced row-echelon matrix with a TI-89 calculator.

that exhibits the solution of the linear system having augmented coefficient matrix $\mathbf{A}$. The same calculation is illustrated in the calculator screen of Fig. 3.3.1. Solve then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. For instance, the element in the second row and third column of $\mathbf{AI}$ is

$$(a_{21})(0) + (a_{22})(0) + (a_{23})(1) = a_{23}.$$

If a is a nonzero real number and $b = a^{-1}$, then $ab = ba = 1$. Given a nonzero square matrix $\mathbf{A}$, the question as to whether there exists an *inverse matrix* $\mathbf{B}$, one such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, is more complicated and is investigated in Section 3.5.

Problems

In Problems 1–4, two matrices **A** *and* **B** *and two numbers c and d are given. Compute the matrix* $c\mathbf{A} + d\mathbf{B}$.

1. $\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 3 & -4 \end{bmatrix}, c = 3, d = 4$

2. $\mathbf{A} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -2 & 3 & 1 \\ 7 & 1 & 5 \end{bmatrix}, c = 5, d = -3$

3. $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix}, c = -2, d = 4$

4. $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & -3 \\ 5 & -2 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 6 & -3 & -4 \\ 5 & 2 & -1 \\ 0 & 7 & 9 \end{bmatrix}, c = 7, d = 5$

In Problems 5–12, two matrices **A** *and* **B** *are given. Calculate whichever of the matrices* **AB** *and* **BA** *is defined.*

5. $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 8 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 7 & -4 & 3 \\ 1 & 5 & -2 \\ 0 & 3 & 9 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$

9. $\mathbf{A} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}$

11. $\mathbf{A} = \begin{bmatrix} 3 & -5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & -7 & 5 \\ 3 & 9 & 10 \end{bmatrix}$

In Problems 13–16, three matrices **A**, **B**, *and* **C** *are given. Verify by computation of both sides the associative law* $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

13. $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

14. $\mathbf{A} = \begin{bmatrix} 2 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

15. $\mathbf{A} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$

16. $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix},$

$\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix}$

In Problems 17–22, first write each given homogeneous system in the matrix form $\mathbf{Ax} = \mathbf{0}$. *Then find the solution in vector form, as in Eq. (9).*

17. $\begin{aligned} x_1 \quad - 5x_3 + 4x_4 &= 0 \\ x_2 + 2x_3 - 7x_4 &= 0 \end{aligned}$

18. $\begin{aligned} x_1 - 3x_2 \quad + 6x_4 &= 0 \\ x_3 + 9x_4 &= 0 \end{aligned}$

19. $\begin{aligned} x_1 \quad + 3x_4 - x_5 &= 0 \\ x_2 \quad - 2x_4 + 6x_5 &= 0 \\ x_3 + x_4 - 8x_5 &= 0 \end{aligned}$

20. $\begin{aligned} x_1 - 3x_2 \quad + 7x_5 &= 0 \\ x_3 \quad - 2x_5 &= 0 \\ x_4 - 10x_5 &= 0 \end{aligned}$

21. $\begin{aligned} x_1 \quad - x_3 + 2x_4 + 7x_5 &= 0 \\ x_2 + 2x_3 - 3x_4 + 4x_5 &= 0 \end{aligned}$

22. $\begin{aligned} x_1 - x_2 \quad + 7x_4 + 3x_5 &= 0 \\ x_3 - x_4 - 2x_5 &= 0 \end{aligned}$

Problems 23 through 26 introduce the idea—developed more fully in the next section—of a multiplicative inverse of a square matrix.

23. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find $\mathbf{B}$ so that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$ as follows: First equate entries on the two sides of the equation $\mathbf{AB} = \mathbf{I}$. Then solve the resulting four equations for a, b, c, and d. Finally verify that $\mathbf{BA} = \mathbf{I}$ as well.

24. Repeat Problem 23, but with $\mathbf{A}$ replaced by the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

25. Repeat Problem 23, but with $\mathbf{A}$ replaced by the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}.$$

26. Use the technique of Problem 23 to show that if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix},$$

then there does *not* exist a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$. *Suggestion*: Show that the system of four equations in a, b, c, and d is inconsistent.

27. A **diagonal matrix** is a square matrix of the form

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

in which every element off the main diagonal is zero. Show that the product $\mathbf{AB}$ of two $n \times n$ diagonal matrices $\mathbf{A}$ and $\mathbf{B}$ is again a diagonal matrix. State a concise rule for quickly computing $\mathbf{AB}$. Is it clear that $\mathbf{AB} = \mathbf{BA}$? Explain.

Problems 28 through 30 develop a method of computing powers of a square matrix.

28. The positive integral powers of a square matrix $\mathbf{A}$ are defined as follows:

$$\mathbf{A}^1 = \mathbf{A}, \quad \mathbf{A}^2 = \mathbf{AA}, \quad \mathbf{A}^3 = \mathbf{AA}^2, \quad \mathbf{A}^4 = \mathbf{AA}^3, \ldots, \quad \mathbf{A}^{n+1} = \mathbf{AA}^n, \ldots.$$

Suppose that r and s are positive integers. Prove that $\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s}$ and that $(\mathbf{A}^r)^s = \mathbf{A}^{rs}$ (in close analogy with the laws of exponents for real numbers).

29. If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then show that

$$\mathbf{A}^2 = (a+d)\mathbf{A} - (ad - bc)\mathbf{I},$$

where $\mathbf{I}$ denotes the 2×2 identity matrix. Thus every 2×2 matrix $\mathbf{A}$ satisfies the equation

$$\mathbf{A}^2 - (\operatorname{trace} \mathbf{A})\mathbf{A} + (\det \mathbf{A})\mathbf{I} = \mathbf{0}$$

where $\det \mathbf{A} = ad - bc$ denotes the determinant of the matrix $\mathbf{A}$, and trace $\mathbf{A}$ denotes the sum of its diagonal elements. This result is the 2-dimensional case of the Cayley-Hamilton theorem of Section 6.3.

30. The formula in Problem 29 can be used to compute $\mathbf{A}^2$ without an explicit matrix multiplication. It follows that

$$\mathbf{A}^3 = (a+d)\mathbf{A}^2 - (ad - bc)\mathbf{A}$$

without an explicit matrix multiplication,

$$\mathbf{A}^4 = (a+d)\mathbf{A}^3 - (ad - bc)\mathbf{A}^2,$$

and so on. Use this method to compute $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{A}^4$, and $\mathbf{A}^5$ given

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problems 31–38 illustrate ways in which the algebra of matrices is not analogous to the algebra of real numbers.

31. **(a)** Suppose that $\mathbf{A}$ and $\mathbf{B}$ are the matrices of Example 5. Show that $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2$.

(b) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property that $\mathbf{AB} = \mathbf{BA}$. Show that $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

32. **(a)** Suppose that $\mathbf{A}$ and $\mathbf{B}$ are the matrices of Example 5. Show that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

(b) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices such that $\mathbf{AB} = \mathbf{BA}$. Show that $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

33. Find four different 2×2 matrices $\mathbf{A}$, with each main diagonal element either $+1$ or -1, such that $\mathbf{A}^2 = \mathbf{I}$.

34. Find a 2×2 matrix $\mathbf{A}$ with each element $+1$ or -1 such that $\mathbf{A}^2 = \mathbf{0}$. The formula of Problem 29 may be helpful.

35. Use the formula of Problem 29 to find a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{I}$ but such that $\mathbf{A}^2 = \mathbf{A}$.

36. Find a 2×2 matrix $\mathbf{A}$ with each main diagonal element zero such that $\mathbf{A}^2 = \mathbf{I}$.

37. Find a 2×2 matrix $\mathbf{A}$ with each main diagonal element zero such that $\mathbf{A}^2 = -\mathbf{I}$.

38. This is a continuation of the previous two problems. Find two nonzero 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}^2 + \mathbf{B}^2 = \mathbf{0}$.

39. Use matrix multiplication to show that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ and c_1 and c_2 are real numbers, then $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution.

40. **(a)** Use matrix multiplication to show that if $\mathbf{x}_0$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_1$ is a solution of the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x}_0 + \mathbf{x}_1$ is also a solution of the nonhomogeneous system.

(b) Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the nonhomogeneous system of part (a). Show that $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

41. This is a continuation of Problem 32. Show that if $\mathbf{A}$ and $\mathbf{B}$ are square matrices such that $\mathbf{AB} = \mathbf{BA}$, then

$$(\mathbf{A} + \mathbf{B})^3 = \mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{AB}^2 + \mathbf{B}^3$$

and

$$(\mathbf{A} + \mathbf{B})^4 = \mathbf{A}^4 + 4\mathbf{A}^3\mathbf{B} + 6\mathbf{A}^2\mathbf{B}^2 + 4\mathbf{AB}^3 + \mathbf{B}^4.$$

42. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{I} + \mathbf{N}.$$

(a) Show that $\mathbf{N}^2 \neq \mathbf{0}$ but $\mathbf{N}^3 = \mathbf{0}$.

(b) Use the binomial formulas of Problem 41 to compute

$$\mathbf{A}^2 = (\mathbf{I}+\mathbf{N})^2 = \mathbf{I} + 2\mathbf{N} + \mathbf{N}^2,$$
$$\mathbf{A}^3 = (\mathbf{I}+\mathbf{N})^3 = \mathbf{I} + 3\mathbf{N} + 3\mathbf{N}^2,$$

and

$$\mathbf{A}^4 = (\mathbf{I}+\mathbf{N})^4 = \mathbf{I} + 4\mathbf{N} + 6\mathbf{N}^2.$$

43. Consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

First verify by direct computation that $\mathbf{A}^2 = 3\mathbf{A}$. Then conclude that $\mathbf{A}^{n+1} = 3^n\mathbf{A}$ for every positive integer n.

44. Let $\mathbf{A} = \begin{bmatrix} a_{hi} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_{ij} \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} c_{jk} \end{bmatrix}$ be matrices of sizes $m \times n$, $n \times p$, and $p \times q$, respectively. To establish the associative law $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, proceed as follows. By Equation (16) the hjth element of $\mathbf{AB}$ is

$$\sum_{i=1}^{n} a_{hi} b_{ij}.$$

By another application of Equation (16), the hkth element of $(\mathbf{AB})\mathbf{C}$ is

$$\sum_{j=1}^{p}\left(\sum_{i=1}^{n} a_{hi} b_{ij}\right) c_{jk} = \sum_{i=1}^{n}\sum_{j=1}^{p} a_{hi} b_{ij} c_{jk}.$$

Show similarly that the double sum on the right is also equal to the hkth element of $\mathbf{A}(\mathbf{BC})$. Hence the $m \times q$ matrices $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are equal.

nverses of Matrices

Recall that the $n \times n$ **identity matrix** is the diagonal matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1}$$

having ones on its main diagonal and zeros elsewhere. It is not difficult to deduce directly from the definition of the matrix product that $\mathbf{I}$ acts like an identity for matrix multiplication:

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IB} = \mathbf{B} \tag{2}$$

if the sizes of $\mathbf{A}$ and $\mathbf{B}$ are such that the products $\mathbf{AI}$ and $\mathbf{IB}$ are defined. It is, nevertheless, instructive to derive the identities in (2) formally from the two basic facts about matrix multiplication that we state below. First, recall that the notation

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{bmatrix} \tag{3}$$

expresses the $m \times n$ matrix $\mathbf{A}$ in terms of its column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$.

Fact 1 $\mathbf{Ax}$ *in terms of columns of* $\mathbf{A}$

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-vector, then

$$\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n. \tag{4}$$

The reason is that when each row vector of $\mathbf{A}$ is multiplied by the column vector $\mathbf{x}$, its jth element is multiplied by x_j.

Fact 2 $\mathbf{AB}$ *in terms of columns of* $\mathbf{B}$

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$ is an $n \times p$ matrix, then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix}. \tag{5}$$

That is, *the jth column of* $\mathbf{AB}$ *is the product of* $\mathbf{A}$ *and the jth column of* $\mathbf{B}$. The reason is that the elements of the jth column of $\mathbf{AB}$ are obtained by multiplying the individual rows of $\mathbf{A}$ by the jth column of $\mathbf{B}$.

3.5 Problems

In Problems 1–8, first apply the formulas in (9) to find $\mathbf{A}^{-1}$*. Then use* $\mathbf{A}^{-1}$ *(as in Example 5) to solve the system* $\mathbf{Ax} = \mathbf{b}$.

1. $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 5 & 12 \\ 7 & 17 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 3 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} 7 & 9 \\ 5 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 8 & 15 \\ 5 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

In Problems 9–22, use the method of Example 7 to find the inverse $\mathbf{A}^{-1}$ *of each given matrix* $\mathbf{A}$.

9. $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 3 \\ 3 & 10 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}$

14. $\begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 4 & 13 \\ 3 & 2 & 12 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 2 \\ 2 & -3 & -3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}$

In Problems 23–28, use the method of Example 8 to find a matrix $\mathbf{X}$ *such that* $\mathbf{AX} = \mathbf{B}$.

23. $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix}$

24. $\mathbf{A} = \begin{bmatrix} 7 & 6 \\ 8 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix}$

25. $\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 2 & 7 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$

26. $\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

27. $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 7 \\ 2 & 2 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

28. $\mathbf{A} = \begin{bmatrix} 6 & 5 & 3 \\ 5 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ -1 & 3 & 5 & 0 \\ 1 & 1 & 0 & 5 \end{bmatrix}$

29. Verify parts (a) and (b) of Theorem 3.

Problems 30 through 37 explore the properties of matrix inverses.

30. Suppose that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are invertible matrices of the same size. Show that the product $\mathbf{ABC}$ is invertible and that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

31. Suppose that $\mathbf{A}$ is an invertible matrix and that r and s are negative integers. Verify that $\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s}$ and that $(\mathbf{A}^r)^s = \mathbf{A}^{rs}$.

32. Prove that if $\mathbf{A}$ is an invertible matrix and $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$. Thus *invertible* matrices can be canceled.

33. Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{Ax} = \mathbf{x}$ for every n-vector $\mathbf{x}$. Show that $\mathbf{A} = \mathbf{I}$.

34. Show that a diagonal matrix is invertible if and only if each diagonal element is nonzero. In this case, state concisely how the inverse matrix is obtained.

35. Let $\mathbf{A}$ be an $n \times n$ matrix with either a row or a column consisting only of zeros. Show that $\mathbf{A}$ is not invertible.

36. Show that $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.

37. Suppose that $ad - bc \neq 0$ and $\mathbf{A}^{-1}$ is defined as in Equation (9). Verify directly that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Problems 38 through 40 explore the effect of multiplying by an elementary matrix.

38. Let $\mathbf{E}$ be the elementary matrix $\mathbf{E}_1$ of Example 6. If $\mathbf{A}$ is a 2×2 matrix, show that $\mathbf{EA}$ is the result of multiplying the first row of $\mathbf{A}$ by 3.

39. Let $\mathbf{E}$ be the elementary matrix $\mathbf{E}_2$ of Example 6 and suppose that $\mathbf{A}$ is a 3×3 matrix. Show that $\mathbf{EA}$ is the result upon adding twice the first row of $\mathbf{A}$ to its third row.

40. Let $\mathbf{E}$ be the elementary matrix $\mathbf{E}_3$ of Example 6. Show that $\mathbf{EA}$ is the result of interchanging the first two rows of the matrix $\mathbf{A}$.

Problems 41 and 42 complete the proof of Eq. (2).

41. Show that the ith row of the product $\mathbf{AB}$ is $\mathbf{A}_i\mathbf{B}$, where $\mathbf{A}_i$ is the ith row of the matrix $\mathbf{A}$.

42. Apply the result of Problem 41 to show that if $\mathbf{B}$ is an $m \times n$ matrix and $\mathbf{I}$ is the $m \times m$ identity matrix, then $\mathbf{IB} = \mathbf{B}$.

43. Suppose that the matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent. Use Theorem 5 to prove that $\mathbf{B} = \mathbf{GA}$, where $\mathbf{G}$ is a product of elementary matrices.

44. Show that every invertible matrix is a product of elementary matrices.

45. Extract from the proof of Theorem 7 a self-contained proof of the following fact: If **A** and **B** are square matrices such that **AB** = **I**, then **A** and **B** are invertible.

46. Deduce from the result of Problem 45 that if **A** and **B** are square matrices whose product **AB** is invertible, then **A** and **B** are themselves invertible.

3.5 Application Automated Solution of Linear Systems

Go to goo.gl/IcSRZK to **download** this application's computing resources including *Maple*/*Mathematica*/MATLAB.

Linear systems with more than two or three equations are most frequently solved with the aid of calculators or computers. If an $n \times n$ linear system is written in the matrix form $\mathbf{Ax} = \mathbf{b}$, then we need to calculate first the inverse matrix $\mathbf{A}^{-1}$ and then the matrix product $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Suppose the $n \times n$ matrix **A** and the column vector **b** have been entered (as illustrated in the 3.2 Application). If **A** is invertible, then the inverse matrix $\mathbf{A}^{-1}$ is calculated by the *Maple* command `with(linalg): inverse(A)`, the *Mathematica* command `Inverse[A]`, or the MATLAB command `inv(A)`. Consequently, the solution vector **x** is calculated by the *Maple* command

```
with(linalg):  x := multiply(inverse(A),b);
```

or the *Mathematica* command

```
x = Inverse[A].b
```

or the MATLAB command

```
x = inv(A)*b
```

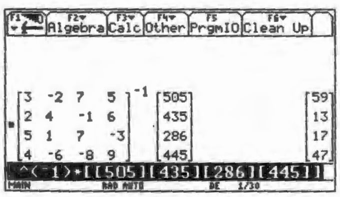

FIGURE 3.5.2. TI-89 solution of a linear system $\mathbf{Ax} = \mathbf{b}$.

Figure 3.5.2 illustrates a similar calculator solution of the linear system

$$\begin{aligned}
3x_1 - 2x_2 + 7x_3 + 5x_4 &= 505\\
2x_1 + 4x_2 - x_3 + 6x_4 &= 435\\
5x_1 + x_2 + 7x_3 - 3x_4 &= 286\\
4x_1 - 6x_2 - 8x_3 + 9x_4 &= 445
\end{aligned}$$

for the solution $x_1 = 59$, $x_2 = 13$, $x_3 = 17$, $x_4 = 47$. This solution is also given by the Wolfram|Alpha query

```
A = ((3, -2, 7, 5), (2, 4, -1, 6), (5, 1, 7, -3),
     (4, -6, -8, 9)),
b = (505, 435, 286, 445),
inv(A).b
```

Remark Whereas the preceding commands illustrate the handy use of conveniently available inverse matrices to solve linear systems, it might be mentioned that modern computer systems employ direct methods—involving Gaussian elimination and still more sophisticated techniques—that are more efficient and numerically reliable to solve a linear system $\mathbf{Ax} = \mathbf{b}$ *without* first calculating the inverse matrix $\mathbf{A}^{-1}$. ∎

Use an available calculator or computer system to solve the linear systems in Problems 1–6 of the 3.3 Application. The applied problems below are elementary in character—resembling the "word problems" of high school algebra—but might illustrate the practical advantages of automated solutions.

1. You are walking down the street minding your own business when you spot a small but heavy leather bag lying on the sidewalk. It turns out to contain U.S. Mint American Eagle gold coins of the following types:

- One-half ounce gold coins that sell for $285 each,
- One-quarter ounce gold coins that sell for $150 each, and

3.6 Problems

Use cofactor expansions to evaluate the determinants in Problems 1–6. Expand along the row or column that minimizes the amount of computation that is required.

1. $\begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 5 & 0 \end{vmatrix}$

2. $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{vmatrix}$

4. $\begin{vmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{vmatrix}$

5. $\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 0 & 11 & -5 & 0 \\ -2 & 4 & 13 & 6 & 5 \\ 0 & 0 & 5 & 0 & 0 \\ 7 & 6 & -9 & 17 & 7 \\ 0 & 0 & 8 & 2 & 0 \end{vmatrix}$

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

7. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{vmatrix}$

9. $\begin{vmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{vmatrix}$

10. $\begin{vmatrix} -3 & 6 & 5 \\ 1 & -2 & -4 \\ 2 & -5 & 12 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 2 & 4 & 6 & 9 \end{vmatrix}$

12. $\begin{vmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{vmatrix}$

Use the method of elimination to evaluate the determinants in Problems 13–20.

13. $\begin{vmatrix} -4 & 4 & -1 \\ -1 & -2 & 2 \\ 1 & 4 & 3 \end{vmatrix}$

14. $\begin{vmatrix} 4 & 2 & -2 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix}$

15. $\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix}$

16. $\begin{vmatrix} 2 & 4 & -2 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix}$

18. $\begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{vmatrix}$

19. $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{vmatrix}$

20. $\begin{vmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 \\ 0 & 1 & -2 & 3 \\ -1 & 4 & -2 & 4 \end{vmatrix}$

Use Cramer's rule to solve the systems in Problems 21–32.

21. $\begin{aligned} 3x + 4y &= 2 \\ 5x + 7y &= 1 \end{aligned}$

22. $\begin{aligned} 5x + 8y &= 3 \\ 8x + 13y &= 5 \end{aligned}$

23. $\begin{aligned} 17x + 7y &= 6 \\ 12x + 5y &= 4 \end{aligned}$

24. $\begin{aligned} 11x + 15y &= 10 \\ 8x + 11y &= 7 \end{aligned}$

25. $\begin{aligned} 5x + 6y &= 12 \\ 3x + 4y &= 6 \end{aligned}$

26. $\begin{aligned} 6x + 7y &= 3 \\ 8x + 9y &= 4 \end{aligned}$

27. $\begin{aligned} 5x_1 + 2x_2 - 2x_3 &= 1 \\ x_1 + 5x_2 - 3x_3 &= -2 \\ 5x_1 - 3x_2 + 5x_3 &= 2 \end{aligned}$

28. $\begin{aligned} 5x_1 + 4x_2 - 2x_3 &= 4 \\ 2x_1 \qquad\quad + 3x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \end{aligned}$

29. $\begin{aligned} 3x_1 - x_2 - 5x_3 &= 3 \\ 4x_1 - 4x_2 - 3x_3 &= -4 \\ x_1 \qquad\quad - 5x_3 &= 2 \end{aligned}$

30. $\begin{aligned} x_1 + 4x_2 + 2x_3 &= 3 \\ 4x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 - 5x_3 &= -3 \end{aligned}$

31. $\begin{aligned} 2x_1 \qquad\quad - 5x_3 &= -3 \\ 4x_1 - 5x_2 + 3x_3 &= 3 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$

32. $\begin{aligned} 3x_1 + 4x_2 - 3x_3 &= 5 \\ 3x_1 - 2x_2 + 4x_3 &= 7 \\ 3x_1 + 2x_2 - x_3 &= 3 \end{aligned}$

Apply Theorem 5 to find the inverse $\mathbf{A}^{-1}$ of each matrix $\mathbf{A}$ given in Problems 33–40.

33. $\begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 2 & 0 & 3 \\ -5 & -4 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

35. $\begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}$

36. $\begin{bmatrix} -4 & 4 & 3 \\ 3 & -1 & -5 \\ 1 & 0 & -5 \end{bmatrix}$

37. $\begin{bmatrix} -4 & 1 & 5 \\ -2 & 4 & 5 \\ -3 & -3 & -1 \end{bmatrix}$

38. $\begin{bmatrix} 3 & 4 & -3 \\ 3 & 2 & -1 \\ -3 & 2 & -4 \end{bmatrix}$

39. $\begin{bmatrix} -3 & -2 & 3 \\ 0 & 3 & 2 \\ 2 & 3 & -5 \end{bmatrix}$

40. $\begin{bmatrix} 2 & 4 & -3 \\ 2 & -3 & -1 \\ -5 & 0 & -3 \end{bmatrix}$

41. Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ if $\mathbf{A}$ and $\mathbf{B}$ are arbitrary 2×2 matrices.

42. Consider the 2×2 matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix},$$

where $\mathbf{x}$ and $\mathbf{y}$ denote the row vectors of $\mathbf{B}$. Then the product $\mathbf{AB}$ can be written in the form

$$\mathbf{AB} = \begin{bmatrix} a\mathbf{x} + b\mathbf{y} \\ c\mathbf{x} + d\mathbf{y} \end{bmatrix}.$$

Use this expression and the properties of determinants to show that

$$\det \mathbf{AB} = (ad - bc) \begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} = (\det \mathbf{A})(\det \mathbf{B}).$$

Thus the determinant of a product of 2×2 matrices is equal to the product of their determinants.

Each of Problems 43–46 lists a special case of one of Property 1 through Property 5. Verify it by expanding the determinant on the left-hand side along an appropriate row or column.

43. $\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

44. $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

45. $\begin{vmatrix} a_1 & b_1 & c_1 + d_1 \\ a_2 & b_2 & c_2 + d_2 \\ a_3 & b_3 & c_3 + d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

46. $\begin{vmatrix} a_{11} + ka_{12} & a_{12} & a_{13} \\ a_{21} + ka_{22} & a_{22} & a_{23} \\ a_{31} + ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Problems 47 through 49 develop properties of matrix transposes.

47. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size. Show that: (a) $(\mathbf{A}^T)^T = \mathbf{A}$; (b) $(c\mathbf{A})^T = c\mathbf{A}^T$; and (c) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

48. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices such that $\mathbf{AB}$ is defined. Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. Begin by recalling that the ijth element of $\mathbf{AB}$ is obtained by multiplying elements in the ith row of $\mathbf{A}$ with those in the jth column of $\mathbf{B}$. What is the ijth element of $\mathbf{B}^T\mathbf{A}^T$?

49. Let $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ be a 3×3 matrix. Show that $\det(\mathbf{A}^T) = \det \mathbf{A}$ by expanding $\det \mathbf{A}$ along its first row and $\det(\mathbf{A}^T)$ along its first column.

50. Suppose that $\mathbf{A}^2 = \mathbf{A}$. Prove that $|\mathbf{A}| = 0$ or $|\mathbf{A}| = 1$.

51. Suppose that $\mathbf{A}^n = \mathbf{0}$ (the zero matrix) for some positive integer n. Prove that $|\mathbf{A}| = 0$.

52. The square matrix $\mathbf{A}$ is called **orthogonal** provided that $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that the determinant of such a matrix must be either $+1$ or -1.

53. The matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be **similar** provided that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ for some invertible matrix $\mathbf{P}$. Show that if $\mathbf{A}$ and $\mathbf{B}$ are similar, then $|\mathbf{A}| = |\mathbf{B}|$.

54. Deduce from Theorems 2 and 3 that if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then $\mathbf{AB}$ is invertible if and only if both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

55. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Suppose it is known that *either* $\mathbf{AB} = \mathbf{I}$ *or* $\mathbf{BA} = \mathbf{I}$. Use the result of Problem 54 to conclude that $\mathbf{B} = \mathbf{A}^{-1}$.

56. Let $\mathbf{A}$ be an $n \times n$ matrix with $\det \mathbf{A} = 1$ and with all elements of $\mathbf{A}$ integers.

(a) Show that $\mathbf{A}^{-1}$ has only integer entries.

(b) Suppose that $\mathbf{b}$ is an n-vector with only integer entries. Show that the solution vector $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$ has only integer entries.

57. Let $\mathbf{A}$ be a 3×3 upper triangular matrix with nonzero determinant. Show by explicit computation that $\mathbf{A}^{-1}$ is also upper triangular.

58. Figure 3.6.2 shows an acute triangle with angles A, B, and C and opposite sides a, b, and c. By dropping a perpendicular from each vertex to the opposite side, derive the equations

$$\begin{aligned} c \cos B + b \cos C &= a \\ c \cos A + a \cos C &= b \\ a \cos B + b \cos A &= c. \end{aligned}$$

Regarding these as linear equations in the unknowns $\cos A$, $\cos B$, and $\cos C$, use Cramer's rule to derive the **law of cosines** by solving for

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Thus

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Note that the case $A = \pi/2$ (90°) reduces to the Pythagorean theorem.

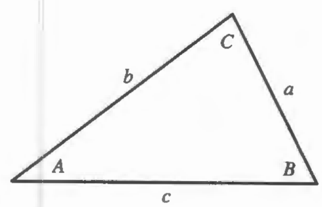

FIGURE 3.6.2. The triangle of Problem 58.

59. Show that

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \quad \text{and} \quad \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4.$$

60. Consider the $n \times n$ determinant

$$B_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{vmatrix}$$

in which each entry on the main diagonal is a 2, each entry on the two adjacent diagonals is a 1, and every other entry is zero.

(a) Expand along the first row to show that

$$B_n = 2B_{n-1} - B_{n-2}.$$

(b) Prove by induction on n that $B_n = n + 1$ for $n \geq 2$.

Problems 61–64 deal with the ***Vandermonde determinant***

$$V(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

that will play an important role in Section 3.7.

61. Show by direct computation that $V(a, b) = b - a$ and that

$$V(a, b, c) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - a)(c - a)(c - b).$$

62. The formulas in Problem 61 are the cases $n = 2$ and $n = 3$ of the general formula

$$V(x_1, x_2, \ldots, x_n) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j). \tag{25}$$

The case $n = 4$ is

$$V(x_1, x_2, x_3, x_4) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \times (x_4 - x_1)(x_4 - x_2)(x_4 - x_3).$$

Prove this as follows. Given x_1, x_2, and x_3, define the cubic polynomial $P(y)$ to be

$$P(y) = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & y & y^2 & y^3 \end{vmatrix}. \tag{26}$$

Because $P(x_1) = P(x_2) = P(x_3) = 0$ (why?), the roots of $P(y)$ are x_1, x_2, and x_3. It follows that

$$P(y) = k(y - x_1)(y - x_2)(y - x_3),$$

where k is the coefficient of y^3 in $P(y)$. Finally, observe that expansion of the 4×4 determinant in (26) along its last row gives $k = V(x_1, x_2, x_3)$ and that $V(x_1, x_2, x_3, x_4) = P(x_4)$.

63. Generalize the argument in Problem 62 to prove the formula in (25) by induction on n. Begin with the $(n - 1)$st-degree polynomial

$$P(y) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \\ 1 & y & y^2 & \cdots & y^{n-1} \end{vmatrix}.$$

64. Use the formula in (25) to evaluate the two determinants given next.

(a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$

ar Equations and Curve Fitting

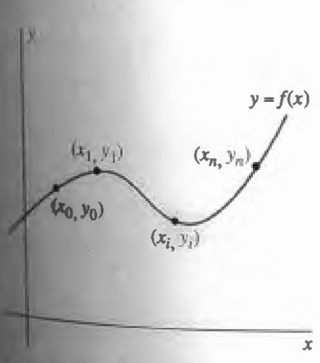

FIGURE 3.7.1. A curve $y = f(x)$ interpolating (that is, passing through) given data points.

Linear algebra has important applications to the common scientific problem of representing empirical data by means of equations or functions of specified types. We give here only a brief introduction to this extensive subject.

Typically, we begin with a collection of given *data points* $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ that are to be represented by a specific type of function $y = f(x)$. For instance, y might be the volume of a sample of gas when its temperature is x. Thus the given data points are the results of experiment or measurement, and we want to determine the curve $y = f(x)$ in the xy-plane so that it passes through each of these points; see Figure 3.7.1. Thus we speak of "fitting" the curve to the data points.

We will confine our attention largely to polynomial curves. A **polynomial of degree** n is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1}$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are constants. The data point (x_i, y_i) lies on the curve $y = f(x)$ provided that $f(x_i) = y_i$. The condition that this be so for each