The remainder of this chapter is devoted largely to techniques for solving a differential equation by first finding the Laplace transform of its solution. It is then vital for us to know that this uniquely determines the solution of the differential equation; that is, that the function of s we have found has only one inverse Laplace transform that could be the desired solution. The following theorem is proved in Chapter 6 of Churchill's *Operational Mathematics*, 3rd ed. (New York: McGraw-Hill, 1972).

THEOREM 3 Uniqueness of Inverse Laplace Transforms

Suppose that the functions $f(t)$ and $g(t)$ satisfy the hypotheses of Theorem 2, so that their Laplace transforms $F(s)$ and $G(s)$ both exist. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, +\infty)$ both f and g are continuous.

Thus two piecewise continuous functions of exponential order with the same Laplace transform can differ only at their isolated points of discontinuity. This is of no importance in most practical applications, so we may regard inverse Laplace transforms as being essentially unique. In particular, two solutions of a differential equation must both be continuous, and hence must be the same solution if they have the same Laplace transform.

Historical Remark Laplace transforms have an interesting history. The integral in the definition of the Laplace transform probably appeared first in the work of Euler. It is customary in mathematics to name a technique or theorem for the next person after Euler to discover it (else there would be several hundred different examples of "Euler's theorem"). In this case, the next person was the French mathematician Pierre Simon de Laplace (1749–1827), who employed such integrals in his work on probability theory. The so-called operational methods for solving differential equations, which are based on Laplace transforms, were not exploited by Laplace. Indeed, they were discovered and popularized by practicing engineers—notably the English electrical engineer Oliver Heaviside (1850–1925). These techniques were successfully and widely applied before they had been rigorously justified, and around the beginning of the twentieth century their validity was the subject of considerable controversy. One reason is that Heaviside blithely assumed the existence of functions whose Laplace transforms contradict the condition that $F(s) \to 0$ as $s \to 0$, thereby raising questions as to the meaning and nature of functions in mathematics. (This is reminiscent of the way Leibniz two centuries earlier had obtained correct results in calculus using "infinitely small" real numbers, thereby raising questions as to the nature and role of numbers in mathematics.)

10.1 Problems

Apply the definition in (1) to find directly the Laplace transforms of the functions described (by formula or graph) in Problems 1 through 10.

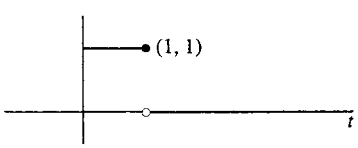
1. $f(t) = t$
2. $f(t) = t^2$
3. $f(t) = e^{3t+1}$
4. $f(t) = \cos t$
5. $f(t) = \sinh t$
6. $f(t) = \sin^2 t$
7. 

FIGURE 10.1.6.

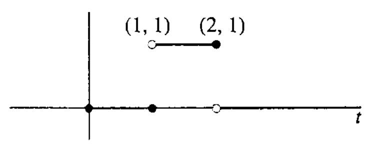
8. 

FIGURE 10.1.7.

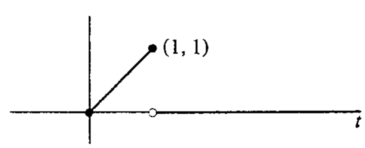
9. 

FIGURE 10.1.8.

10.

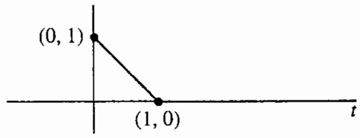


FIGURE 10.1.9.

Use the transforms in Fig. 10.1.2 to find the Laplace transforms of the functions in Problems 11 through 22. A preliminary integration by parts may be necessary.

11. $f(t) = \sqrt{t} + 3t$

12. $f(t) = 3t^{5/2} - 4t^3$

13. $f(t) = t - 2e^{3t}$

14. $f(t) = t^{3/2} - e^{-10t}$

15. $f(t) = 1 + \cosh 5t$

16. $f(t) = \sin 2t + \cos 2t$

17. $f(t) = \cos^2 2t$

18. $f(t) = \sin 3t \cos 3t$

19. $f(t) = (1 + t)^3$

20. $f(t) = te^t$

21. $f(t) = t \cos 2t$

22. $f(t) = \sinh^2 3t$

Use the transforms in Fig. 10.1.2 to find the inverse Laplace transforms of the functions in Problems 23 through 32.

23. $F(s) = \dfrac{3}{s^4}$

24. $F(s) = s^{-3/2}$

25. $F(s) = \dfrac{1}{s} - \dfrac{2}{s^{5/2}}$

26. $F(s) = \dfrac{1}{s+5}$

27. $F(s) = \dfrac{3}{s-4}$

28. $F(s) = \dfrac{3s+1}{s^2+4}$

29. $F(s) = \dfrac{5-3s}{s^2+9}$

30. $F(s) = \dfrac{9+s}{4-s^2}$

31. $F(s) = \dfrac{10s-3}{25-s^2}$

32. $F(s) = 2s^{-1}e^{-3s}$

33. Derive the transform of $f(t) = \sin kt$ by the method used in the text to derive the formula in (16).

34. Derive the transform of $f(t) = \sinh kt$ by the method used in the text to derive the formula in (14).

35. Use the tabulated integral

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$$

to obtain $\mathcal{L}\{\cos kt\}$ directly from the definition of the Laplace transform.

36. Show that the function $f(t) = \sin(e^{t^2})$ is of exponential order as $t \to +\infty$ but that its derivative is not.

37. Given $a > 0$, let $f(t) = 1$ if $0 \leqq t < a$, $f(t) = 0$ if $t \geqq a$. First, sketch the graph of the function f, making clear its value at $t = a$. Then express f in terms of unit step functions to show that $\mathcal{L}\{f(t)\} = s^{-1}(1 - e^{-as})$.

38. Given that $0 < a < b$, let $f(t) = 1$ if $a \leqq t < b$, $f(t) = 0$ if either $t < a$ or $t \geqq b$. First, sketch the graph of the function f, making clear its values at $t = a$ and $t = b$. Then express f in terms of unit step functions to show that $\mathcal{L}\{f(t)\} = s^{-1}(e^{-as} - e^{-bs})$.

39. The *unit staircase function* is defined as follows:

$$f(t) = n \quad \text{if} \quad n - 1 \leqq t < n, \quad n = 1, 2, 3, \ldots$$

(a) Sketch the graph of f to see why its name is appropriate. **(b)** Show that

$$f(t) = \sum_{n=0}^{\infty} u(t - n)$$

for all $t \geqq 0$. **(c)** Assume that the Laplace transform of the infinite series in part (b) can be taken termwise (it can). Apply the geometric series to obtain the result

$$\mathcal{L}\{f(t)\} = \frac{1}{s(1 - e^{-s})}.$$

40. **(a)** The graph of the function f is shown in Fig. 10.1.10. Show that f can be written in the form

$$f(t) = \sum_{n=0}^{\infty} (-1)^n u(t - n).$$

(b) Use the method of Problem 39 to show that

$$\mathcal{L}\{f(t)\} = \frac{1}{s(1 + e^{-s})}.$$

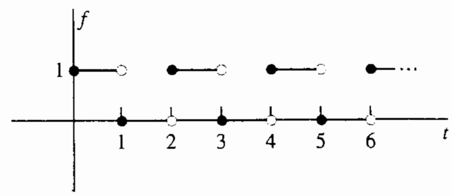


FIGURE 10.1.10. The graph of the function of Problem 40.

41. The graph of the *square-wave function* $g(t)$ is shown in Fig. 10.1.11. Express g in terms of the function f of Problem 40 and hence deduce that

$$\mathcal{L}\{g(t)\} = \frac{1 - e^{-s}}{s(1 + e^{-s})} = \frac{1}{s} \tanh \frac{s}{2}.$$

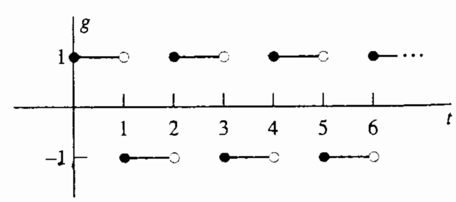


FIGURE 10.1.11. The graph of the function of Problem 41.

42. Given constants a and b, define $h(t)$ for $t \geqq 0$ by

$$h(t) = \begin{cases} a & \text{if } n - 1 \leqq t < n \text{ and } n \text{ is odd;} \\ b & \text{if } n - 1 \leqq t < n \text{ and } n \text{ is even.} \end{cases}$$

Sketch the graph of h and apply one of the preceding problems to show that

$$\mathcal{L}\{h(t)\} = \frac{a + be^{-s}}{s(1 + e^{-s})}.$$

that may not agree with the actual values $f(t_{n-1})$ and $f(t_n)$. The value of an integral on an interval is not affected by changing the values of the integrand at the endpoints. However, if the fundamental theorem of calculus is applied to find the value of the integral, then the antiderivative function must be continuous on the closed interval. We therefore use the "continuous from within the interval" endpoint values above in evaluating (by parts) the integrals on the right in (19). The result is

$$\begin{aligned}\sum_{n=1}^{k}\Big[e^{-st}f(t)\Big]_{t_{n-1}}^{t_n} &= \Big[-f(t_0^+) + e^{-st_1}f(t_1^-)\Big] + \Big[-e^{-st_1}f(t_1^+) + e^{-st_2}f(t_2^-)\Big] \\ &\quad + \cdots + \Big[-e^{st_{k-2}}f(t_{k-2}^+) + e^{-st_{k-1}}f(t_{k-1}^-)\Big] \\ &\quad + \Big[-e^{st_{k-1}}f(t_{k-1}^+) + e^{-st_k}f(t_k^-)\Big] \\ &= -f(0^+) - \sum_{n=1}^{k-1} j_f(t_n) + e^{-sb}f(b^-), \end{aligned} \tag{20'}$$

where

$$j_f(t_n) = f(t_n^+) - f(t_n^-) \tag{21}$$

denotes the (finite) *jump* in $f(t)$ at $t = t_n$. Assuming that $\mathcal{L}\{f'(t)\}$ exists, we therefore get the generalization

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^+) - \sum_{n=1}^{\infty} e^{-st_n} j_f(t_n) \tag{22}$$

of $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ when we now take the limit in (19) as $b \to +\infty$.

Example 7 Let $f(t) = 1 + [\![t]\!]$ be the unit staircase function; its graph is shown in Fig. 10.2.8. Then $f(0) = 1$, $f'(t) \equiv 0$, and $j_f(n) = 1$ for each integer $n = 1, 2, 3, \ldots$. Hence Eq. (22) yields

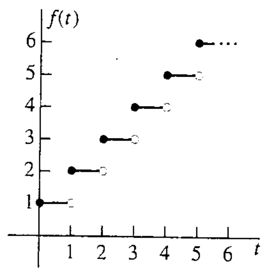


FIGURE 10.2.8. The graph of the unit staircase function of Example 7.

$$0 = sF(s) - 1 - \sum_{n=1}^{\infty} e^{-ns},$$

so the Laplace transform of $f(t)$ is

$$F(s) = \frac{1}{s}\sum_{n=0}^{\infty} e^{-ns} = \frac{1}{s(1-e^{-s})}.$$

In the last step we used the formula for the sum of a geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

with $x = e^{-s} < 1$. ■

10.2 Problems

Use Laplace transforms to solve the initial value problems in Problems 1 through 16.

1. $x'' + 4x = 0$; $x(0) = 5$, $x'(0) = 0$
2. $x'' + 9x = 0$; $x(0) = 3$, $x'(0) = 4$
3. $x'' - x' - 2x = 0$; $x(0) = 0$, $x'(0) = 2$
4. $x'' + 8x' + 15x = 0$; $x(0) = 2$, $x'(0) = -3$
5. $x'' + x = \sin 2t$; $x(0) = 0 = x'(0)$
6. $x'' + 4x = \cos t$; $x(0) = 0 = x'(0)$
7. $x'' + x = \cos 3t$; $x(0) = 1$, $x'(0) = 0$
8. $x'' + 9x = 1$; $x(0) = 0 = x'(0)$

9. $x'' + 4x' + 3x = 1;\ x(0) = 0 = x'(0)$

10. $x'' + 3x' + 2x = t;\ x(0) = 0,\ x'(0) = 2$

11. $x' = 2x + y,\ y' = 6x + 3y;\ x(0) = 1,\ y(0) = -2$

12. $x' = x + 2y,\ y' = x + e^{-t};\ x(0) = y(0) = 0$

13. $x' + 2y' + x = 0,\ x' - y' + y = 0;\ x(0) = 0,\ y(0) = 1$

14. $x'' + 2x + 4y = 0,\ y'' + x + 2y = 0;\ x(0) = y(0) = 0,\ x'(0) = y'(0) = -1$

15. $x'' + x' + y' + 2x - y = 0,\ y'' + x' + y' + 4x - 2y = 0;\ x(0) = y(0) = 1,\ x'(0) = y'(0) = 0$

16. $x' = x + z,\ y' = x + y,\ z' = -2x - z;\ x(0) = 1,\ y(0) = 0,\ z(0) = 0$

Apply Theorem 2 to find the inverse Laplace transforms of the functions in Problems 17 through 24.

17. $F(s) = \dfrac{1}{s(s-3)}$

18. $F(s) = \dfrac{3}{s(s+5)}$

19. $F(s) = \dfrac{1}{s(s^2+4)}$

20. $F(s) = \dfrac{2s+1}{s(s^2+9)}$

21. $F(s) = \dfrac{1}{s^2(s^2+1)}$

22. $F(s) = \dfrac{1}{s(s^2-9)}$

23. $F(s) = \dfrac{1}{s^2(s^2-1)}$

24. $F(s) = \dfrac{1}{s(s+1)(s+2)}$

25. Apply Theorem 1 to derive $\mathcal{L}\{\sin kt\}$ from the formula for $\mathcal{L}\{\cos kt\}$.

26. Apply Theorem 1 to derive $\mathcal{L}\{\cosh kt\}$ from the formula for $\mathcal{L}\{\sinh kt\}$.

27. **(a)** Apply Theorem 1 to show that

$$\mathcal{L}\{t^n e^{at}\} = \frac{n}{s-a}\mathcal{L}\{t^{n-1}e^{at}\}.$$

(b) Deduce that $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$ for $n = 1, 2, 3, \dots$.

Apply Theorem 1 as in Example 5 to derive the Laplace transforms in Problems 28 through 30.

28. $\mathcal{L}\{t\cos kt\} = \dfrac{s^2-k^2}{(s^2+k^2)^2}$

29. $\mathcal{L}\{t\sinh kt\} = \dfrac{2ks}{(s^2-k^2)^2}$

30. $\mathcal{L}\{t\cosh kt\} = \dfrac{s^2+k^2}{(s^2-k^2)^2}$

31. Apply the results in Example 5 and Problem 28 to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^3}(\sin kt - kt\cos kt).$$

Apply the extension of Theorem 1 in Eq. (22) to derive the Laplace transforms given in Problems 32 through 37.

32. $\mathcal{L}\{u(t-a)\} = s^{-1}e^{-as}$ for $a > 0$.

33. If $f(t) = 1$ on the interval $[a, b]$ (where $0 < a < b$) and $f(t) = 0$ otherwise, then

$$\mathcal{L}\{f(t)\} = \frac{e^{-as} - e^{-bs}}{s}.$$

34. If $f(t) = (-1)^{[\![t]\!]}$ is the square-wave function whose graph is shown in Fig. 10.2.9, then

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\tanh\frac{s}{2}.$$

(*Suggestion:* Use the geometric series.)

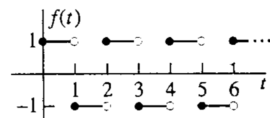


FIGURE 10.2.9. The graph of the square-wave function of Problem 34.

35. If $f(t)$ is the *unit on–off function* whose graph is shown in Fig. 10.2.10, then

$$\mathcal{L}\{f(t)\} = \frac{1}{s(1+e^{-s})}.$$

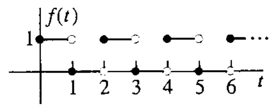


FIGURE 10.2.10. The graph of the on–off function of Problem 35.

36. If $g(t)$ is the *triangular wave function* whose graph is shown in Fig. 10.2.11, then

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2}\tanh\frac{s}{2}.$$

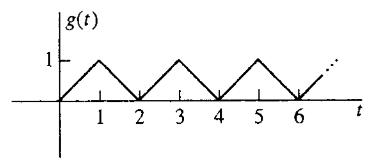


FIGURE 10.2.11. The graph of the triangular wave function of Problem 36.

37. If $f(t)$ is the *sawtooth function* whose graph is shown in Fig. 10.2.12, then

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.$$

(*Suggestion:* Note that $f'(t) \equiv 1$ where it is defined.)

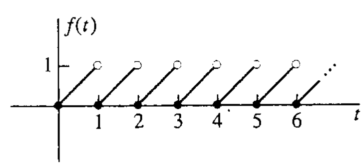


FIGURE 10.2.12. The graph of the sawtooth function of Problem 37.

Solution First we observe that

$$\mathcal{L}\{y''(t)\} = s^2Y(s), \quad \mathcal{L}\{y^{(4)}(t)\} = s^4Y(s), \quad \text{and} \quad \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}.$$

Hence the transformed equation is

$$(s^4 + 2s^2 + 1)Y(s) = \frac{4}{(s-1)^2}.$$

Thus our problem is to find the inverse transform of

$$\begin{aligned} Y(s) &= \frac{4}{(s-1)^2(s^2+1)^2} \\ &= \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{s^2+1}. \end{aligned} \tag{19}$$

If we multiply by the common denominator $(s-1)^2(s^2+1)^2$, we get the equation

$$\begin{aligned} &A(s^2+1)^2 + B(s-1)(s^2+1)^2 + Cs(s-1)^2 \\ &\quad + D(s-1)^2 + Es(s-1)^2(s^2+1) + F(s-1)^2(s^2+1) = 4. \end{aligned} \tag{20}$$

Upon substituting $s = 1$ we find that $A = 1$.

Equation (20) is an identity that holds for all values of s. To find the values of the remaining coefficients, we substitute in succession the values $s = 0$, $s = -1$, $s = 2$, $s = -2$, and $s = 3$ in Eq. (20). This yields the system

$$\begin{aligned} -B \qquad\quad + D \qquad\quad + F &= 3, \\ -8B - 4C + 4D - 8E + 8F &= 0, \\ 25B + 2C + D + 10E + 5F &= -21, \\ -75B - 18C + 9D - 90E + 45F &= -21, \\ 200B + 12C + 4D + 120E + 40F &= -96 \end{aligned} \tag{21}$$

of five linear equations in B, C, D, E, and F. With the aid of a calculator programmed to solve linear systems, we find that $B = -2$, $C = 2$, $D = 0$, $E = 2$, and $F = 1$.

We now substitute in Eq. (19) the coefficients we have found, and thus obtain

$$Y(s) = \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{2s}{(s^2+1)^2} + \frac{2s+1}{s^2+1}.$$

Recalling Eq. (16), the translation property, and the familiar transforms of $\cos t$ and $\sin t$, we see finally that the solution of the given initial value problem is

$$y(t) = (t-2)e^t + (t+1)\sin t + 2\cos t.$$ ■

10.3 Problems

Apply the translation theorem to find the Laplace transforms of the functions in Problems 1 through 4.

1. $f(t) = t^4 e^{\pi t}$

2. $f(t) = t^{3/2} e^{-4t}$

3. $f(t) = e^{-2t} \sin 3\pi t$

4. $f(t) = e^{-t/2} \cos 2\left(t - \frac{1}{8}\pi\right)$

Apply the translation theorem to find the inverse Laplace transforms of the functions in Problems 5 through 10.

5. $F(s) = \dfrac{3}{2s-4}$

6. $F(s) = \dfrac{s-1}{(s+1)^3}$

7. $F(s) = \dfrac{1}{s^2+4s+4}$

8. $F(s) = \dfrac{s+2}{s^2+4s+5}$

9. $F(s) = \dfrac{3s+5}{s^2-6s+25}$

10. $F(s) = \dfrac{2s-3}{9s^2-12s+20}$

Use partial fractions to find the inverse Laplace transforms of the functions in Problems 11 through 22.

11. $F(s) = \dfrac{1}{s^2-4}$

12. $F(s) = \dfrac{5s-6}{s^2-3s}$

13. $F(s) = \dfrac{5-2s}{s^2+7s+10}$

14. $F(s) = \dfrac{5s-4}{s^3-s^2-2s}$

15. $F(s) = \dfrac{1}{s^3 - 5s^2}$ **16.** $F(s) = \dfrac{1}{(s^2 + s - 6)^2}$

17. $F(s) = \dfrac{1}{s^4 - 16}$ **18.** $F(s) = \dfrac{s^3}{(s-4)^4}$

19. $F(s) = \dfrac{s^2 - 2s}{s^4 + 5s^2 + 4}$ **20.** $F(s) = \dfrac{1}{s^4 - 8s^2 + 16}$

21. $F(s) = \dfrac{s^2 + 3}{(s^2 + 2s + 2)^2}$ **22.** $F(s) = \dfrac{2s^3 - s^2}{(4s^2 - 4s + 5)^2}$

Use the factorization

$$s^4 + 4a^4 = (s^2 - 2as + 2a^2)(s^2 + 2as + 2a^2)$$

to derive the inverse Laplace transforms listed in Problems 23 through 26.

23. $\mathcal{L}^{-1}\left\{\dfrac{s^3}{s^4 + 4a^4}\right\} = \cosh at \cos at$

24. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^4 + 4a^4}\right\} = \dfrac{1}{2a^2} \sinh at \sin at$

25. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{s^4 + 4a^4}\right\} = \dfrac{1}{2a}(\cosh at \sin at + \sinh at \cos at)$

26. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4 + 4a^4}\right\} = \dfrac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$

Use Laplace transforms to solve the initial value problems in Problems 27 through 38.

27. $x'' + 6x' + 25x = 0;\ x(0) = 2,\ x'(0) = 3$

28. $x'' - 6x' + 8x = 2;\ x(0) = x'(0) = 0$

29. $x'' - 4x = 3t;\ x(0) = x'(0) = 0$

30. $x'' + 4x' + 8x = e^{-t};\ x(0) = x'(0) = 0$

31. $x^{(3)} + x'' - 6x' = 0;\ x(0) = 0,\ x'(0) = x''(0) = 1$

32. $x^{(4)} - x = 0;\ x(0) = 1,\ x'(0) = x''(0) = x^{(3)}(0) = 0$

33. $x^{(4)} + x = 0;\ x(0) = x'(0) = x''(0) = 0,\ x^{(3)}(0) = 1$

34. $x^{(4)} + 13x'' + 36x = 0;\ x(0) = x''(0) = 0,\ x'(0) = 2,\ x^{(3)}(0) = -13$

35. $x^{(4)} + 8x'' + 16x = 0;\ x(0) = x'(0) = x''(0) = 0,\ x^{(3)}(0) = 1$

36. $x^{(4)} + 2x'' + x = e^{2t};\ x(0) = x'(0) = x''(0) = x^{(3)}(0) = 0$

37. $x'' + 4x' + 13x = te^{-t};\ x(0) = 0,\ x'(0) = 2$

38. $x'' + 6x' + 18x = \cos 2t;\ x(0) = 1,\ x'(0) = -1$

Resonance

Problems 39 and 40 illustrate two types of resonance in a mass–spring–dashpot system with given external force $F(t)$ and with the initial conditions $x(0) = x'(0) = 0$.

39. Suppose that $m = 1$, $k = 9$, $c = 0$, and $F(t) = 6\cos 3t$. Use the inverse transform given in Eq. (16) to derive the solution $x(t) = t \sin 3t$. Construct a figure that illustrates the resonance that occurs.

40. Suppose that $m = 1$, $k = 9.04$, $c = 0.4$, and $F(t) = 6e^{-t/5}\cos 3t$. Derive the solution

$$x(t) = te^{-t/5} \sin 3t.$$

Show that the maximum value of the amplitude function $A(t) = te^{-t/5}$ is $A(5) = 5/e$. Thus (as indicated in Fig. 10.3.5) the oscillations of the mass increase in amplitude during the first 5 s before being damped out as $t \to +\infty$.

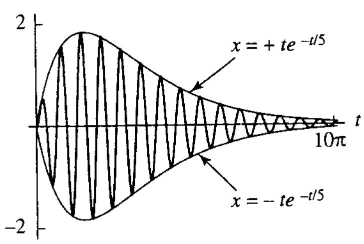


FIGURE 10.3.5. The graph of the damped oscillation in Problem 40.

10.3 Application Damping and Resonance Investigations

Go to goo.gl/4ntUK9 to download this application's computing resources including *Maple*/*Mathematica*/MATLAB.

Here we outline a *Maple* investigation of the behavior of the mass–spring–dashpot system

$$mx'' + cx' + kx = F(t), \quad x(0) = x'(0) = 0 \tag{1}$$

with parameter values

```
m := 25;    c := 10;    k := 226;
```

in response to a variety of possible external forces:

1. $F(t) \equiv 226$

This should give damped oscillations "leveling off" to a constant solution (why?).

2. $F(t) = 901 \cos 3t$

So integration of $F(\sigma)$ from s to $+\infty$ gives

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \left(\int_0^\infty e^{-\sigma t} f(t)\,dt\right) d\sigma.$$

Under the hypotheses of the theorem, the order of integration may be reversed (see Churchill's book once again); it follows that

$$\begin{aligned}\int_s^\infty F(\sigma)\,d\sigma &= \int_0^\infty \left(\int_s^\infty e^{-\sigma t} f(t)\,d\sigma\right) dt \\ &= \int_0^\infty \left[\frac{e^{-\sigma t}}{-t}\right]_{\sigma=s}^{\infty} f(t)\,dt \\ &= \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}.\end{aligned}$$

This verifies Eq. (12), and Eq. (13) follows upon first applying $\mathcal{L}^{-1}$ and then multiplying by t. ▲

10.4 Problems

Find the convolution $f(t) * g(t)$ in Problems 1 through 6.

1. $f(t) = t,\ g(t) \equiv 1$
2. $f(t) = t,\ g(t) = e^{at}$
3. $f(t) = g(t) = \sin t$
4. $f(t) = t^2,\ g(t) = \cos t$
5. $f(t) = g(t) = e^{at}$
6. $f(t) = e^{at},\ g(t) = e^{bt}\quad (a \neq b)$

Apply the convolution theorem to find the inverse Laplace transforms of the functions in Problems 7 through 14.

7. $F(s) = \dfrac{1}{s(s-3)}$
8. $F(s) = \dfrac{1}{s(s^2+4)}$
9. $F(s) = \dfrac{1}{(s^2+9)^2}$
10. $F(s) = \dfrac{1}{s^2(s^2+k^2)}$
11. $F(s) = \dfrac{s^2}{(s^2+4)^2}$
12. $F(s) = \dfrac{1}{s(s^2+4s+5)}$
13. $F(s) = \dfrac{s}{(s-3)(s^2+1)}$
14. $F(s) = \dfrac{s}{s^4+5s^2+4}$

In Problems 15 through 22, apply either Theorem 2 or Theorem 3 to find the Laplace transform of $f(t)$.

15. $f(t) = t\sin 3t$
16. $f(t) = t^2\cos 2t$
17. $f(t) = te^{2t}\cos 3t$
18. $f(t) = te^{-t}\sin^2 t$
19. $f(t) = \dfrac{\sin t}{t}$
20. $f(t) = \dfrac{1-\cos 2t}{t}$
21. $f(t) = \dfrac{e^{3t}-1}{t}$
22. $f(t) = \dfrac{e^t - e^{-t}}{t}$

Find the inverse transforms of the functions in Problems 23 through 28.

23. $F(s) = \ln\dfrac{s-2}{s+2}$
24. $F(s) = \ln\dfrac{s^2+1}{s^2+4}$
25. $F(s) = \ln\dfrac{s^2+1}{(s+2)(s-3)}$
26. $F(s) = \tan^{-1}\dfrac{3}{s+2}$
27. $F(s) = \ln\left(1+\dfrac{1}{s^2}\right)$
28. $F(s) = \dfrac{s}{(s^2+1)^3}$

In Problems 29 through 34, transform the given differential equation to find a nontrivial solution such that $x(0) = 0$.

29. $tx'' + (t-2)x' + x = 0$
30. $tx'' + (3t-1)x' + 3x = 0$
31. $tx'' - (4t+1)x' + 2(2t+1)x = 0$
32. $tx'' + 2(t-1)x' - 2x = 0$
33. $tx'' - 2x' + tx = 0$
34. $tx'' + (4t-2)x' + (13t-4)x = 0$
35. Apply the convolution theorem to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du = e^t \operatorname{erf}\sqrt{t}.$$

(*Suggestion:* Substitute $u = \sqrt{t}$.)

In Problems 36 through 38, apply the convolution theorem to derive the indicated solution $x(t)$ of the given differential equation with initial conditions $x(0) = x'(0) = 0$.

36. $x'' + 4x = f(t);\ x(t) = \dfrac{1}{2}\displaystyle\int_0^t f(t-\tau)\sin 2\tau\,d\tau$
37. $x'' + 2x' + x = f(t);\ x(t) = \displaystyle\int_0^t \tau e^{-\tau} f(t-\tau)\,d\tau$
38. $x'' + 4x' + 13x = f(t);$

$$x(t) = \frac{1}{3}\int_0^t f(t-\tau)e^{-2\tau}\sin 3\tau\,d\tau$$

Termwise Inverse Transformation of Series

In Chapter 2 of Churchill's Operational Mathematics, *the following theorem is proved. Suppose that $f(t)$ is continuous for $t \geqq 0$, that $f(t)$ is of exponential order as $t \to +\infty$, and that*

$$F(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+k+1}}$$

where $0 \leqq k < 1$ and the series converges absolutely for $s > c$. Then

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n t^{n+k}}{\Gamma(n+k+1)}.$$

Apply this result in Problems 39 through 41.

39. In Example 5 it was shown that

$$\mathcal{L}\{J_0(t)\} = \frac{C}{\sqrt{s^2+1}} = \frac{C}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2}.$$

Expand with the aid of the binomial series and then compute the inverse transformation term by term to obtain

$$J_0(t) = C \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Finally, note that $J_0(0) = 1$ implies that $C = 1$.

40. Expand the function $F(s) = s^{-1/2}e^{-1/s}$ in powers of s^{-1} to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}e^{-1/s}\right\} = \frac{1}{\sqrt{\pi t}}\cos 2\sqrt{t}.$$

41. Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-1/s}\right\} = J_0\left(2\sqrt{t}\right).$$

10.5 Periodic and Piecewise Continuous Input Functions

Mathematical models of mechanical or electrical systems often involve functions with discontinuities corresponding to external forces that are turned abruptly on or off. One such simple on–off function is the unit step function that we introduced in Section 10.1. Recall that the *unit step function* at $t = a$ is defined by

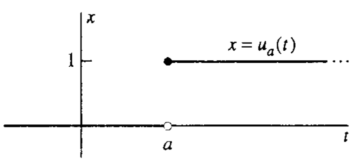


FIGURE 10.5.1. The graph of the unit step function at $t = a$.

$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geqq a. \end{cases} \tag{1}$$

The notation $u_a(t)$ indicates succinctly where the unit upward step in value takes place (Fig. 10.5.1), whereas $u(t-a)$ connotes the sometimes useful idea of a "time delay" a before the step is made.

In Example 8 of Section 10.1 we saw that if $a \geqq 0$, then

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}. \tag{2}$$

Because $\mathcal{L}\{u(t)\} = 1/s$, Eq. (2) implies that multiplication of the transform of $u(t)$ by e^{-as} corresponds to the translation $t \to t - a$ in the original independent variable. Theorem 1 tells us that this fact, when properly interpreted, is a general property of the Laplace transformation.

THEOREM 1 Translation on the t-Axis

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \tag{3a}$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \tag{3b}$$

for $s > c + a$.

Note that

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a, \\ f(t-a) & \text{if } t \geqq a. \end{cases} \tag{4}$$