Math3280 exam 1 **Name** **Score**

1. Identify the type and obtain the solutions of the following first order equations:

(a) $\frac{dy}{dx} = \frac{e^x}{y}$.

separable

$$\int y\,dy = \int e^x\,dx + c$$

$$\frac{1}{2}y^2 = e^x + C$$

$$\boxed{y = \pm\sqrt{2e^x + c}}$$

(b) $x\frac{dy}{dx} - 2y = x^3\cos x, \quad y(\frac{\pi}{2}) = 1.$

linear

$p(x) = -\frac{2}{x}, \quad Q(x) = x^2\cos x$

$$I(x) = e^{-\int \frac{2}{x}dx} = x^{-2}$$

$$y = x^2\left(\int x^{-2}\cdot x^2\cos x\,dx + c\right) = x^2(\sin x + c)$$

$$1 = \left(\frac{\pi}{2}\right)^2(1 + c) \Rightarrow c = \left(\frac{2}{\pi}\right)^2 - 1$$

$$\boxed{y = x^2\left(\sin x + \frac{4}{\pi^2} - 1\right)}$$

(c) $x^2\frac{dy}{dx} + 2xy = 5y^3$.

$p(x) = \frac{2}{x} \quad Q(x) = \frac{5}{x^2} \quad n = 3$ Bernoulli

Let $v = y^{-2}$, we have linear eqn

$$\frac{dv}{dx} + (-2)\cdot\frac{2}{x}\cdot v = (-2)\frac{5}{x^2}$$

$$I(x) = e^{\int \frac{-4}{x}dx} = x^{-4}$$

$$v = x^4\left(\int x^{-4}\cdot\frac{-10}{x^2}dx + c\right) = x^4\left(+\frac{2}{x^5} + c\right)$$

$$= cx^4 + \frac{2}{x}$$

$$y = \frac{1}{\pm\sqrt{v}} = \cancel{\frac{1}{x^2\left(\frac{-10}{x} + c\right)^{1/2}}}$$

$$\boxed{y = \frac{1}{\pm\sqrt{cx^4 + \frac{2}{x}}}}$$

(d) $(x^3 + \frac{y}{x})\,dx + (y^2 + \ln x)\,dy = 0.$

$M = x^3 + \frac{y}{x} \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x}$

$N = y^2 + \ln x \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{x}$ > exact

$F = \int M\,dx + g'(y) = \frac{1}{4}x^4 + y\ln x + g'(y)$

$g'(y) = N - \frac{\partial}{\partial y}\int M\,dx = y^2 + \ln x - \ln x = y^2 \Rightarrow g(y) = \frac{1}{3}y^3$

soln:

$$\boxed{\frac{1}{4}x^4 + y\ln x + \frac{1}{3}y^3 = C}$$ ✓

(e) $\frac{dy}{dx} = (2x + y - 1)^2.$

linear coeffs.

$V = 2x + y - 1 \Rightarrow \frac{dv}{dx} = 2 + \frac{dy}{dx}$

new eqn: $\frac{dv}{dx} - 2 = v^2 \Rightarrow \int \frac{1}{2+v^2}\,dv = \int dx$

$\Rightarrow \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right) = x + c \Rightarrow \frac{v}{\sqrt{2}} = \tan(\sqrt{2}x + c) \Rightarrow v = \sqrt{2}\tan(\sqrt{2}x + c)$

$$\boxed{y = v - 2x + 1 = \sqrt{2}\tan(\sqrt{2}x + c) - 2x + 1}$$

(f) $x^2\frac{dy}{dx} = xy + y^2.$

homogeneous

$\frac{dy}{dx} = \frac{xy + y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$

let $v = \frac{y}{x} \Rightarrow$ new eqn $v + x\frac{dv}{dx} = \cancel{v} + v^2$

$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx \Rightarrow -\frac{1}{v} = \ln|x| + c \Rightarrow v = \frac{-1}{\ln|x| + c}$

$$\boxed{y = \frac{-x}{\ln|x| + c}}$$

2. Suppose the death rate of a population $P(t)$ at time t is $3P(t)$ and the birth rate is $2P(t)$. It is known that the initial population is 100. Set up a DE and solve it for $P(t)$.

$$\begin{cases} \frac{dP}{dt} = 2P - 3P \\ P(0) = 100 \end{cases} \Rightarrow \quad P(t) = P(0)\, e^{-t}$$

$$\boxed{P(t) = 100\, e^{-t}}$$

3. Solve $xy'' = y'$.

Let $V = \frac{dy}{dx} \Rightarrow$ new eqn $x \frac{dv}{dx} = v \Rightarrow \int \frac{1}{v}\, dv = \int \frac{1}{x}\, dx + c$

$\Rightarrow \ln|V| = \ln|x| + C \Rightarrow V = C_1 x$

Then $\frac{dy}{dx} = C_1 x \Rightarrow \boxed{y = C_3 x^2 + C_2}$

4. Solve $yy'' = 3(y')^2$.

Let $V = \frac{dy}{dx} \Rightarrow$ new eqn $y \cdot V \cdot \frac{dv}{dy} = 3V^2$

$\Rightarrow \int \frac{1}{v}\, dv = \int \frac{3}{y}\, dy + c \Rightarrow \ln|V| = 3\ln|y| + C$

$\Rightarrow V = C_1 y^3$

Then $\frac{dy}{dx} = C_1 y^3 \Rightarrow \int \frac{dy}{y^3} = \int C_1\, dx$

$\Rightarrow -\frac{1}{2} y^{-2} = C_1 x + C_2 \Rightarrow y^{-2} = C_3 x + C_4 \Rightarrow \boxed{y = \frac{1}{\sqrt{C_3 x + C_4}}}$

5. Given a DE $x\frac{dy}{dx} - 4x^2y + 2y\ln y = 0$. Use subtitution $v = \ln y$. Convert the equation to a new DE of v.

$$V = \ln y \Rightarrow \frac{dv}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$$

divide eqn by y

$$x \cdot \frac{1}{y}\frac{dy}{dx} - 4x^2 + 2\ln y = 0$$

$$\Rightarrow \boxed{x\frac{dv}{dx} - 4x^2 + 2V = 0}$$ linear !

6. The acceleration dv/dt of a Lamborghini is proportional to the differece between 250 km/h and the velocity v of the car. Set a DE for velocity v. DO NOT SOLVE !!!

$$\frac{dv}{dt} = k(250 - v)$$

90 - 100	12

mean	75
median	80.5