Math3280 exam 2 **Name** **Score**

1. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices. Suppose $\det(\mathbf{A}) = 3$ and $\det(\mathbf{B}) = 5$. Are the following statements true or false?

   (a) $\det(\mathbf{AB}) = 15$. True. $\det(AB) = \det(A) \cdot \det(B)$

   (b) $\det(\mathbf{A} + \mathbf{B}) = 8$. False $\det(A+B) \neq \det(A) + \det(B)$

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}.$$

   (a) Find $\mathbf{A}^{-1}$ by elementary row operation.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -1/2 & 3/4 \\ 1/2 & -1/4 \end{bmatrix}$$

   (b) Find $det(\mathbf{A})$.

$$\det(A) = 1 \cdot 2 - 2 \cdot 6 = -4$$

   (c) Find $det(2\mathbf{A})$.

$$\det(2A) = \begin{vmatrix} 2 & 6 \\ 4 & 4 \end{vmatrix} = -16$$

   (d) Find the product matrix $\mathbf{AA}$.

$$A \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$$

   (e) Find the sum $\mathbf{A} + 2\mathbf{A}$.

$$A + 2A = 3A = \begin{bmatrix} 3 & 9 \\ 6 & 6 \end{bmatrix}$$

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3. Given the linear system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & -4 \\ 3 & 6 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solve it by the Cramer's rule.

$$\det(A) = 14 + 18 - 24 - 21 + 24 - 12 = -1$$

$$\det A_1 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 7 & -4 \\ 3 & 6 & 2 \end{vmatrix} = 14 + 12 - 24 - 21 + 24 - 8 = -3$$

$$\det A_2 = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & -4 \\ 3 & 3 & 2 \end{vmatrix} = 4 + 9 - 12 - 6 - 6 + 12 = 1$$

$$\det A_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 7 & 2 \\ 3 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 7 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

By Cramer's rule

$$\begin{cases} x_1 = \dfrac{\det A_1}{\det A} = 3 \\ x_2 = \dfrac{\det A_2}{\det A} = -1 \\ x_3 = \dfrac{\det A_3}{\det A} = 0 \end{cases}$$

Soln: $\vec{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

4. Solve the problem 3 by Gauss elimination method. Write out the reduced Echlon matrix.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & -4 & 2 \\ 3 & 6 & 2 & 3 \end{bmatrix} \xrightarrow[R_1\cdot(-3)+R_3]{R_1\cdot(-3)+R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -7 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[\substack{R_3(7)+R_2 \\ R_3+R_1}]{R_3\cdot(-1)} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_1]{R_2(-1)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Reduced Echlon matrix

$x_1 = 3, \quad x_2 = -1, \quad x_3 = 0.$

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5. Find $\mathbf{A}^{-1}$ for the $\mathbf{A}$ in problem 3. Then find the solution by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ again.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 7 & -4 & 0 & 1 & 0 \\ 3 & 6 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -3 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 2 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 18 & 1 & -7 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 1 & 0 & 0 & -38 & -2 & 15 \\ 0 & 1 & 0 & 18 & 1 & -7 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -38 & -2 & 15 \\ 18 & 1 & -7 \\ 3 & 0 & -1 \end{bmatrix} \qquad X = A^{-1}b = \begin{bmatrix} -38 & -2 & 15 \\ 18 & 1 & -7 \\ 3 & 0 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

6. Write out the definition of linear dependence of vectors $\mathbf{v_1}, \cdots, \mathbf{v_k}$. Then, show $\mathbf{v_1} = (2, 0, 1)$, $\mathbf{v_2} = (-3, 1, -1)$ and $\mathbf{v_3} = (0, -2, -1)$ are linearly dependent.

If $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ ~~has non~~ holds for $c_1 \cdots, c_k$ not all zero. then $\vec{v}_1 \cdots \vec{v}_k$ is linearly dependent

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ has nonzero soln} \Leftrightarrow \begin{vmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Actually $\begin{vmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} = -6 + 6 = 0$

7. Suppose $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent. Prove that $\mathbf{v_1} + \mathbf{v_2}$, $\mathbf{v_1} - \mathbf{v_2}$, and $\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3}$ are also linearly independent.

let $c_1(v_1 + v_2) + c_2(v_1 - v_2) + c_3(v_1 + 2v_2 - v_3) = 0$

$\Rightarrow (c_1 + c_2 + c_3)v_1 + (c_1 - c_2 + 2c_3)v_2 - c_3 v_3 = 0$

Since v_1, v_2, v_3 are linearly independent, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 - c_2 + 2c_3 = 0 \\ c_3 = 0 \end{cases} \Rightarrow \text{only soln } c_1 = c_2 = c_3 = 0.$$

Thus $v_1 + v_2$, $v_1 - v_2$, and $v_1 + 2v_2 - v_3$ are linearly independent.

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8. Write out the definition of a basis for a vector space V. Show $\mathbf{v_1} = (1,2)^T$, $\mathbf{v_2} = (2,1)^T$ is a basis for R^2.

def: $v_1, \dots, v_k$ is a basis for V if (1) they span V (2) they are linearly independent

Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be any vector in R^2. Then pick $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

This means $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ span V.

Since $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \Leftrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ linearly indep.

9. Write the vector $\mathbf{w} = (4,5,6)^T$ as a linear combination of $\mathbf{v_1} = (2,-1,4)^T$, $\mathbf{v_2} = (3,0,1)^T$, and $\mathbf{v_3} = (1,2,-1)^T$ if possible. If not, show it is impossible.

This is to ask whether we can find c_1, c_2, c_3 such that $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & 1 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 0 & 2 & 5 \\ 4 & 1 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 9 \\ -1 & 0 & 2 & 5 \\ 4 & 1 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 3 & 5 & 14 \\ 0 & -11 & -13 & -30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -5 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus $3v_1 - 2v_2 + 4v_3 = w$

10. Let $\mathbf{W}$ be a subset in $\mathbf{R}^4$ such that $x_1 = x_3$ and $x_2 = X_4$. Verify that $\mathbf{W}$ is a subspace of $\mathbf{R}^4$.

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in R^4 \;\middle|\; x_2 = x_4,\ x_1 = x_3 \right\}$$

① pick $\vec{y}, \vec{z} \in W$. $\vec{y} + \vec{z} = \begin{pmatrix} y_1 + z_1 \\ y_2 + z_2 \\ y_3 + z_3 \\ y_4 + z_4 \end{pmatrix}$ where $y_1 + z_1 = y_3 + z_3$ since $y_1 = y_3$, $z_1 = z_3$

$y_2 + z_2 = y_4 + z_4$ since $y_2 = y_4$, $z_2 = z_4$

Thus $\vec{y} + \vec{z} \in W$

② $k\vec{y} = \begin{pmatrix} ky_1 \\ ky_2 \\ ky_3 \\ ky_4 \end{pmatrix}$ where $ky_1 = ky_3$, $ky_2 = ky_4$ since $y_1 = y_3$ $y_2 = y_4$.

Thus $k\vec{y} \in W$ for any constant k.

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11. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Find (i) eigenvalues (ii) eigenspaces associated with each eigenvalue. Give a basis for each eigenspace.

$$(1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = 4$$

for $\lambda_1 = -1$, $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for any c_1, basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

for $\lambda_2 = 4$ $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$

basis $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

12. Given

$$x_1 + 3x_2 + 3x_3 + 3x_4 = 0,$$
$$2x_1 + 7x_2 + 5x_3 - x_4 = 0.$$

Find a basis for the solution space of the above homogeneous system.

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 24 \\ 0 & 1 & -1 & -7 \end{bmatrix}$$

$$\vec{x} = \begin{pmatrix} -6x_3 - 24x_4 \\ x_3 + 7x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -24 \\ 7 \\ 0 \\ 1 \end{pmatrix}$$

Soln space is span$\left\{\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -24 \\ 7 \\ 0 \\ 1 \end{pmatrix}\right\}$, basis $\left\{\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -24 \\ 7 \\ 0 \\ 1 \end{pmatrix}\right\}$

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