Math3280 exam 3 **Name** **Score**

1. Find the general solution of

$$y'' - 5y' + 4y = 0.$$

$r^2 - 5r + 4 = (r - 4)(r - 1) = 0. \quad r_1 = 1, \quad r_2 = 4$

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{4x}}$$

2. Find the solution of the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad r_1 = r_2 = 2$

$y(x) = c_1 e^{2x} + c_2 x e^{2x} \Rightarrow y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$

$\begin{cases} 1 = y(0) = c_1 e^0 + 0 \\ -1 = y'(0) = 2c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -3 \end{cases}$

$$\boxed{y(x) = e^{2x} - 3xe^{2x}}$$

3. Find the general solution of

$$y'' - 4y' + 5y = 0.$$

$r^2 - 4r + 5 = 0 \qquad r_{1,2} = 2 \pm i$

$$\boxed{y(x) = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x}$$

4. Find the general solution of

$$y'' - 5y' + 4y = e^{2t}.$$

The homogeneous solution is from problem 1.

Let $y_p = Ae^{2t} \Rightarrow y_p' = 2Ae^{2t}, \quad y_p'' = 4Ae^{2t}$

plug into the eqn. $4Ae^{2t} - 5(2Ae^{2t}) + 4Ae^{2t} = e^{2t} \Rightarrow -2A = 1, \; A = -\frac{1}{2}$

$y_p(t) = \frac{1}{2}e^{2t}$

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{4t} - \frac{1}{2}e^{2t}}$$

5. Use the variation parameter method to find the general solution of

$$y'' + 4y = 1$$

$r^2+4=0 \Rightarrow y_c = C_1\cos 2x + C_2 \sin 2x$

$W[y_1, y_2] = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$ $\quad y_p = u_1(x)y_1 + u_2(x)y_2$

$u_1(x) = \int \frac{-y_2 f}{W[y_1,y_2]}dx = \int -\frac{1}{2}\sin 2x\, dx = \frac{1}{4}\cos 2x$

$u_2(x) = \int \frac{y_1 f}{W[y_1,y_2]}dx = \int \frac{\cos 2x}{2}dx = \frac{1}{4}\sin 2x$

$\Rightarrow y_p = \frac{1}{4}\cos^2 2x + \frac{1}{4}\sin^2 2x = \frac{1}{4}$

$\boxed{y(x) = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}}$

6. In problem 4, if we replace the nonhomogeneous term e^{2t} by e^{4t}, find y_p.

$y_p = Ate^{4t}$, $y_p' = A(e^{4t} + 4te^{4t})$, $y_p'' = A(4e^{4t} + 4e^{4t} + 16te^{4t})$

plug into the eqn.

$(8Ae^{4t} + 16Ate^{4t}) - 5A(e^{4t} + 4te^{4t}) + 4Ate^{4t} = e^{4t}$

$3Ae^{4t} = e^{4t} \Rightarrow A = \frac{1}{3}$

$\boxed{y_p = \frac{1}{3}te^{4t}}$

7. Show $y_1 = 1$, $y_2 = x$, $y_3 = x^2$ are three linearly independent solutions to the differential equation $y''' = 0$.

$r^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 0$

$y_1 = e^{0x} = 1$. $y_2 = xe^{0x} = x$, $y_3 = x^2 e^{0\cdot x} = x^2$

$W[y_1, y_2, y_3] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0 \Rightarrow y_1, y_2, y_3$ are L.I.

8. Find the general solution of the linear first order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}.$$

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = (9-\lambda)(-2-\lambda) + 30 = \lambda^2 - 7\lambda + 12 = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = 4$$

for $\lambda_1 = 3$,

$$\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 6u_1 + 5u_2 = 0 \Rightarrow \vec{u}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} -5 \\ 6 \end{pmatrix} e^{3t}$$

for $\lambda_2 = 4$

$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 + u_2 = 0 \Rightarrow \vec{u}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

$$\boxed{\vec{X}(t) = C_1 \vec{X}_1 + C_2 \vec{X}_2 = \begin{pmatrix} -5C_1 e^{3t} - C_2 e^{4t} \\ 6C_1 e^{3t} + C_2 e^{4t} \end{pmatrix}}$$

9. Find the general solution of the linear first order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}.$$

$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & +3-\lambda \end{vmatrix} = \overset{\lambda^2 - 9}{\cancel{(-3-\lambda)}} + 18 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 9u_1 + 3(1-i)u_2 = 0 \Rightarrow \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1+i \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{X}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}\cos 3t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 3t = \begin{pmatrix} -\cos 3t - \sin 3t \\ 3\cos 3t \end{pmatrix}$$

$$\vec{X}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 3t + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\sin 3t = \begin{pmatrix} \cos 3t - \sin 3t \\ 3\sin 3t \end{pmatrix}$$

$$\vec{X}(t) = C_1\begin{pmatrix} -\cos 3t - \sin 3t \\ 3\cos 3t \end{pmatrix} + C_2\begin{pmatrix} \cos 3t - \sin 3t \\ 3\sin 3t \end{pmatrix}$$

10. Convert the third order equation

$$y''' + 2y'' - 3y' + y = 0$$

into a first order system. Here the prime means derivative with respect to t.

Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, we get

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 + 3y_2 - 2y_3 \end{cases}$$

11. Given an Euler equation

$$x^2y'' + 2xy' - 6y = 0.$$

Let $y = x^r$ be solution. Plug it into the equation to derive a characteristic equation as we did for the constant coefficient equation. Solve the characteristic equation to get two root r_1, r_2. Then $y_1 = x^{r_1}, y_2 = x^{r_2}$ are two solutions to the Euler equation. Show they are linearly independent. Then write out the general solution.

$y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$. plug into the eqn.

$r(r-1)x^r + 2rx^r - 6x^r = 0 \Rightarrow r(r-1) + 2r - 6 = 0 \Rightarrow r^2 + r - 6 = 0$

$r_1 = -3$. $r_2 = 2 \Rightarrow y_1 = x^{-3}$, $y_2 = x^2$ are two solution. $W[y_1, y_2] = \begin{vmatrix} x^{-3} & x^2 \\ -3x^{-4} & 2x \end{vmatrix}$

$= 5x^{-2} \neq 0$

$$\boxed{y(x) = C_1x^{-3} + C_2x^2}$$

12. For problem 8, add a nonhomogeneous term $\mathbf{f}(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t$. Find a particular solution $\mathbf{x_p}(t)$.

Let $\vec{X}_p = \vec{a}e^t$ plug into the eqn $(\vec{a}e^t)' = A(\vec{a}e^t) + \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t$

$\Rightarrow \vec{a} = A\vec{a} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow \begin{bmatrix} 8 & 5 \\ -6 & -3 \end{bmatrix} \vec{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\Rightarrow \vec{a} = \begin{bmatrix} 8 & 5 \\ -6 & -3 \end{bmatrix}^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{bmatrix} -3 & -5 \\ 6 & 8 \end{bmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -4 \\ 10 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 5/3 \end{pmatrix}$

$$\boxed{\vec{X}_p = \begin{pmatrix} -\frac{2}{3} \\ 5/3 \end{pmatrix} e^t}$$