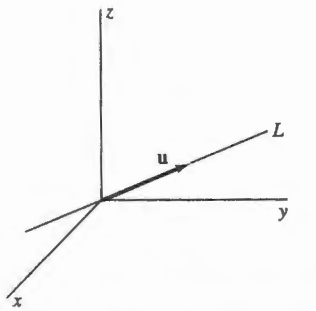

FIGURE 4.1.8. The line L spanned by the vector $\mathbf{u}$.

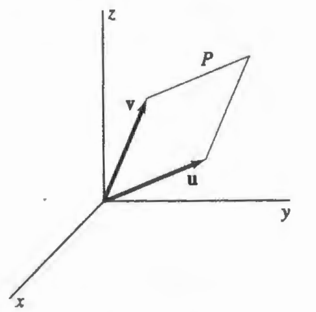

FIGURE 4.1.9. The plane P spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Now we want to show that the proper subspaces of $\mathbf{R}^3$ are what we customarily call lines and planes through the origin. Let V be a subspace of $\mathbf{R}^3$ that is neither $\{\mathbf{0}\}$ nor $\mathbf{R}^3$ itself. There are two cases to consider, depending on whether or not V contains two linearly independent vectors.

CASE 1: Suppose that V does *not* contain two linearly independent vectors. If $\mathbf{u}$ is a fixed nonzero vector in V, then, by condition (ii) above, every scalar multiple $c\mathbf{u}$ is also in V. Conversely, if $\mathbf{v}$ is any other vector in V, then $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, so it follows that $\mathbf{v} = c\mathbf{u}$ for some scalar c. Thus the subspace V is the set of all scalar multiples of the fixed nonzero vector $\mathbf{u}$ and is therefore what we call a **line** through the origin in $\mathbf{R}^3$. (See Fig. 4.1.8.)

CASE 2: Suppose that V contains two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$. It then follows from conditions (i) and (ii) that V contains every linear combination $a\mathbf{u} + b\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$. (See Problem 38.) Conversely, let $\mathbf{w}$ be any other vector in V. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ were linearly independent, then, by Theorem 4, V would be all of $\mathbf{R}^3$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent, so it follows that there exist scalars a and b such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. (See Problem 40.) Thus the subspace V is the set of all linear combinations $a\mathbf{u} + b\mathbf{v}$ of the two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ and is therefore what we call a **plane** through the origin in $\mathbf{R}^3$. (See Fig. 4.1.9.)

Subspaces of the coordinate plane $\mathbf{R}^2$ are defined similarly—they are the nonempty subsets of $\mathbf{R}^2$ that are closed under addition and multiplication by scalars. In Problem 39 we ask you to show that every proper subspace of $\mathbf{R}^2$ is a line through the origin.

Example 5 Let V be the set of all vectors (x, y) in $\mathbf{R}^2$ such that $y = x$. Given $\mathbf{u}$ and $\mathbf{v}$ in V, we may write $\mathbf{u} = (u, u)$ and $\mathbf{v} = (v, v)$. Then $\mathbf{u} + \mathbf{v} = (u + v, u + v)$ and $c\mathbf{u} = (cu, cu)$ are in V. It follows that V is a subspace of $\mathbf{R}^2$. ∎

Example 6 Let V be the set of all vectors (x, y) in $\mathbf{R}^2$ such that $x + y = 1$. Thus V is the straight line that passes through the unit points on the x- and y-axes. Then $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$ are in V, but the vector $\mathbf{u} + \mathbf{v} = (1, 1)$ is not. It follows that V is *not* a subspace of $\mathbf{R}^2$. ∎

Example 6 illustrates the fact that lines that do not pass through the origin are not subspaces of $\mathbf{R}^2$. Because every subspace must contain the zero vector (per Problem 37), only lines and planes that pass through the origin are subspaces of $\mathbf{R}^3$.

4.1 Problems

In Problems 1–4, find $|\mathbf{a} - \mathbf{b}|$, $2\mathbf{a} + \mathbf{b}$, *and* $3\mathbf{a} - 4\mathbf{b}$.

1. $\mathbf{a} = (2, 5, -4)$, $\mathbf{b} = (1, -2, -3)$
2. $\mathbf{a} = (-1, 0, 2)$, $\mathbf{b} = (3, 4, -5)$
3. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$
4. $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{j} - 3\mathbf{k}$

In Problems 5–8, determine whether the given vectors $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent or linearly independent.*

5. $\mathbf{u} = (0, 2)$, $\mathbf{v} = (0, 3)$
6. $\mathbf{u} = (0, 2)$, $\mathbf{v} = (3, 0)$
7. $\mathbf{u} = (2, 2)$, $\mathbf{v} = (2, -2)$
8. $\mathbf{u} = (2, -2)$, $\mathbf{v} = (-2, 2)$

In Problems 9–14, express $\mathbf{w}$ *as a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$.

9. $\mathbf{u} = (1, -2)$, $\mathbf{v} = (-1, 3)$, $\mathbf{w} = (1, 0)$
10. $\mathbf{u} = (3, 4)$, $\mathbf{v} = (2, 3)$, $\mathbf{w} = (0, -1)$
11. $\mathbf{u} = (5, 7)$, $\mathbf{v} = (2, 3)$, $\mathbf{w} = (1, 1)$
12. $\mathbf{u} = (4, 1)$, $\mathbf{v} = (-2, -1)$, $\mathbf{w} = (2, -2)$
13. $\mathbf{u} = (7, 5)$, $\mathbf{v} = (3, 4)$, $\mathbf{w} = (5, -2)$
14. $\mathbf{u} = (5, -2)$, $\mathbf{v} = (-6, 4)$, $\mathbf{w} = (5, 6)$

In Problems 15–18, apply Theorem 4 (that is, calculate a determinant) to determine whether the given vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are linearly dependent or independent.*

15. $\mathbf{u} = (3, -1, 2)$, $\mathbf{v} = (5, 4, -6)$, $\mathbf{w} = (8, 3, -4)$
16. $\mathbf{u} = (5, -2, 4)$, $\mathbf{v} = (2, -3, 5)$, $\mathbf{w} = (4, 5, -7)$
17. $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (3, 0, 1)$, $\mathbf{w} = (1, -2, 2)$
18. $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (4, 3, 1)$, $\mathbf{w} = (3, -2, -4)$

In Problems 19–24, use the method of Example 3 to determine whether the given vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are linearly independent* *or dependent. If they are linearly dependent, find scalars a, b, and c not all zero such that* $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

19. $\mathbf{u} = (2, 0, 1)$, $\mathbf{v} = (-3, 1, -1)$, $\mathbf{w} = (0, -2, -1)$
20. $\mathbf{u} = (5, 5, 4)$, $\mathbf{v} = (2, 3, 1)$, $\mathbf{w} = (4, 1, 5)$
21. $\mathbf{u} = (1, 1, -2)$, $\mathbf{v} = (-2, -1, 6)$, $\mathbf{w} = (3, 7, 2)$
22. $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (5, 1, 3)$, $\mathbf{w} = (0, 1, 2)$
23. $\mathbf{u} = (2, 0, 3)$, $\mathbf{v} = (5, 4, -2)$, $\mathbf{w} = (2, -1, 1)$
24. $\mathbf{u} = (1, 4, 5)$, $\mathbf{v} = (4, 2, 5)$, $\mathbf{w} = (-3, 3, -1)$

In Problems 25–28, express the vector $\mathbf{t}$ *as a linear combination of the vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$.

25. $\mathbf{t} = (2, -7, 9)$, $\mathbf{u} = (1, -2, 2)$, $\mathbf{v} = (3, 0, 1)$, $\mathbf{w} = (1, -1, 2)$
26. $\mathbf{t} = (5, 30, -21)$, $\mathbf{u} = (5, 2, -2)$, $\mathbf{v} = (1, 5, -3)$, $\mathbf{w} = (5, -3, 4)$
27. $\mathbf{t} = (0, 0, 19)$, $\mathbf{u} = (1, 4, 3)$, $\mathbf{v} = (-1, -2, 2)$, $\mathbf{w} = (4, 4, 1)$
28. $\mathbf{t} = (7, 7, 7)$, $\mathbf{u} = (2, 5, 3)$, $\mathbf{v} = (4, 1, -1)$, $\mathbf{w} = (1, 1, 5)$

In Problems 29–32, show that the given set V is closed under addition and under multiplication by scalars and is therefore a subspace of $\mathbf{R}^3$.

29. V is the set of all (x, y, z) such that $x = 0$.
30. V is the set of all (x, y, z) such that $x + y + z = 0$.
31. V is the set of all (x, y, z) such that $2x = 3y$.
32. V is the set of all (x, y, z) such that $z = 2x + 3y$.

In Problems 33–36, show that the given set V is not a subspace of $\mathbf{R}^3$.

33. V is the set of all (x, y, z) such that $y = 1$.
34. V is the set of all (x, y, z) such that $x + y + z = 3$.
35. V is the set of all (x, y, z) such that $z \geq 0$.
36. V is the set of all (x, y, z) such that $xyz = 1$.
37. Show that every subspace V of $\mathbf{R}^3$ contains the zero vector $\mathbf{0}$.
38. Suppose that V is a subspace of $\mathbf{R}^3$. Show that V is closed under the operation of taking linear combinations of pairs of vectors. That is, show that if $\mathbf{u}$ and $\mathbf{v}$ are in V and a and b are scalars, then $a\mathbf{u} + b\mathbf{v}$ is in V.
39. Suppose that V is a proper subspace of $\mathbf{R}^2$ and that $\mathbf{u}$ is a nonzero vector in V. Show that V is the set of all scalar multiples of $\mathbf{u}$ and therefore that V is a line through the origin.
40. Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbf{R}^3$ such that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent but $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent. Show that there exist scalars a and b such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.
41. Let V_1 and V_2 be subspaces of $\mathbf{R}^3$. Their *intersection* $V = V_1 \cap V_2$ is the set of all vectors that lie both in V_1 and in V_2. Show that V is a subspace of $\mathbf{R}^3$.

4.2 The Vector Space $\mathbf{R}^n$ and Subspaces

In Section 4.1 we defined 3-dimensional space $\mathbf{R}^3$ to be the set of all triples (x, y, z) of real numbers. This definition provides a *mathematical model* of the physical space in which we live, because geometric intuition and experience require that the location of every point be specified uniquely by *three* coordinates.

In science fiction, the fourth dimension often plays a rather exotic role. But there are common and ordinary situations where it is convenient to use four (or even more) coordinates rather than just two or three. For example, suppose we want to describe the motion of two points P and Q that are moving in the plane $\mathbf{R}^2$ under the action of some given physical law. (See Fig. 4.2.1.) In order to tell where P and Q are at a given instant, we need to give two coordinates for P and two coordinates for Q. So let us write $P(x_1, x_2)$ and $Q(x_3, x_4)$ to indicate these *four* coordinates. Then the two points P and Q determine a quadruple or 4-tuple (x_1, x_2, x_3, x_4) of real numbers, and any such 4-tuple determines a possible pair of locations of P and Q. In this way the set of all pairs of points P and Q in the plane corresponds to the set of all 4-tuples of real numbers. By analogy with our definition of $\mathbf{R}^3$, we may define *4-dimensional space* $\mathbf{R}^4$ to be the set of all such 4-tuples (x_1, x_2, x_3, x_4). Then we can specify a pair of points P and Q in $\mathbf{R}^2$ by specifying a single *point* (x_1, x_2, x_3, x_4) in $\mathbf{R}^4$, and this viewpoint may simplify our analysis of the motions of the original points P and Q. For instance, it may turn out that their coordinates satisfy some single equation such as

$$3x_1 - 4x_2 + 2x_3 - 5x_4 = 0$$

that is better understood in terms of a single point in $\mathbf{R}^4$ than in terms of the separate points P and Q in $\mathbf{R}^2$. Finally, note that in this example the fourth dimension is quite tangible—it refers simply to the second coordinate x_4 of the point Q.

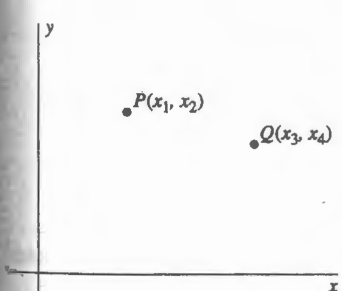

FIGURE 4.2.1. Two points $P(x_1, x_2)$ and $Q(x_3, x_4)$ in $\mathbf{R}^2$.

The reduced echelon form of the coefficient matrix of this system is

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence x_1 and x_2 are the leading variables and x_3 and x_4 are free variables. Back substitution yields the general solution

$$x_3 = s, \quad x_4 = t, \quad x_2 = 4s - 3t, \quad x_1 = 3s + 2t$$

in terms of arbitrary parameters s and t. Thus a typical solution vector of the system in (5) has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that the solution space of the system in (5) can be described as the set of all linear combinations of the form

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \tag{6}$$

where $\mathbf{u} = (3, 4, 1, 0)$ and $\mathbf{v} = (2, -3, 0, 1)$. Thus we have found two particular solution vectors $\mathbf{u}$ and $\mathbf{v}$ of our system that completely determine its solution space [by the formula in (6)]. ■

4.2 Problems

In Problems 1–14, a subset W of some n-space $\mathbf{R}^n$ is defined by means of a given condition imposed on the typical vector $(x_1, x_2, \ldots, x_n)$. Apply Theorem 1 to determine whether or not W is a subspace of $\mathbf{R}^n$.

1. W is the set of all vectors in $\mathbf{R}^3$ such that $x_3 = 0$.
2. W is the set of all vectors in $\mathbf{R}^3$ such that $x_1 = 5x_2$.
3. W is the set of all vectors in $\mathbf{R}^3$ such that $x_2 = 1$.
4. W is the set of all vectors in $\mathbf{R}^3$ such that $x_1 + x_2 + x_3 = 1$.
5. W is the set of all vectors in $\mathbf{R}^4$ such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.
6. W is the set of all vectors in $\mathbf{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$.
7. W is the set of all vectors in $\mathbf{R}^2$ such that $|x_1| = |x_2|$.
8. W is the set of all vectors in $\mathbf{R}^2$ such that $(x_1)^2 + (x_2)^2 = 0$.
9. W is the set of all vectors in $\mathbf{R}^2$ such that $(x_1)^2 + (x_2)^2 = 1$.
10. W is the set of all vectors in $\mathbf{R}^2$ such that $|x_1| + |x_2| = 1$.
11. W is the set of all vectors in $\mathbf{R}^4$ such that $x_1 + x_2 = x_3 + x_4$.
12. W is the set of all vectors in $\mathbf{R}^4$ such that $x_1 x_2 = x_3 x_4$.
13. W is the set of all vectors in $\mathbf{R}^4$ such that $x_1 x_2 x_3 x_4 = 0$.
14. W is the set of all those vectors in $\mathbf{R}^4$ whose components are all nonzero.

In Problems 15–18, apply the method of Example 5 to find two solution vectors $\mathbf{u}$ and $\mathbf{v}$ such that the solution space is the set of all linear combinations of the form $s\mathbf{u} + t\mathbf{v}$.

15. $$\begin{aligned} x_1 - 4x_2 + x_3 - 4x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 0 \\ x_1 + x_2 + x_3 + 6x_4 &= 0 \end{aligned}$$

16. $$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

17. $$\begin{aligned} x_1 + 3x_2 + 8x_3 - x_4 &= 0 \\ x_1 - 3x_2 - 10x_3 + 5x_5 &= 0 \\ x_1 + 4x_2 + 11x_3 - 2x_4 &= 0 \end{aligned}$$

18. $$\begin{aligned} x_1 + 3x_2 + 2x_3 + 5x_4 - x_5 &= 0 \\ 2x_1 + 7x_2 + 4x_3 + 11x_4 + 2x_5 &= 0 \\ 2x_1 + 6x_2 + 5x_3 + 12x_4 - 7x_5 &= 0 \end{aligned}$$

In Problems 19–22, reduce the given system to echelon form to find a single solution vector $\mathbf{u}$ such that the solution space is the set of all scalar multiples of $\mathbf{u}$.

19. $$\begin{aligned} x_1 - 3x_2 - 5x_3 - 6x_4 &= 0 \\ 2x_1 + x_2 + 4x_3 - 4x_4 &= 0 \\ x_1 + 3x_2 + 7x_3 + x_4 &= 0 \end{aligned}$$

20. $$\begin{aligned} x_1 + 5x_2 + x_3 - 8x_4 &= 0 \\ 2x_1 + 5x_2 \qquad - 5x_4 &= 0 \\ 2x_1 + 7x_2 + x_3 - 9x_4 &= 0 \end{aligned}$$

21. $$\begin{aligned} x_1 + 7x_2 + 2x_3 - 3x_4 &= 0 \\ 2x_1 + 7x_2 + x_3 - 4x_4 &= 0 \\ 3x_1 + 5x_2 - x_3 - 5x_4 &= 0 \end{aligned}$$

22. $$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0 \\ 2x_1 + 7x_2 + 5x_3 - x_4 &= 0 \\ 2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0 \end{aligned}$$

23. Show that every subspace W of a vector space V contains the zero vector $\mathbf{0}$.
24. Apply the properties of a vector space V to show each of the following.
    (a) $0\mathbf{u} = \mathbf{0}$ for every $\mathbf{u}$ in V.
    (b) $c\mathbf{0} = \mathbf{0}$ for every scalar c.

(c) $(-1)\mathbf{u} = -\mathbf{u}$ for every $\mathbf{u}$ in V.

Do *not* assume that the vectors in V are n-tuples of real numbers.

25. Show that the nonempty subset W of a vector space V is a subspace of V if and only if for every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in W and every pair of scalars a and b, $a\mathbf{u} + b\mathbf{v}$ is also in W.

26. *Prove*: If $\mathbf{u}$ is a (fixed) vector in the vector space V, then the set W of all scalar multiples $c\mathbf{u}$ of $\mathbf{u}$ is a subspace of V.

27. Let $\mathbf{u}$ and $\mathbf{v}$ be (fixed) vectors in the vector space V. Show that the set W of all linear combinations $a\mathbf{u} + b\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is a subspace of V.

28. Suppose that $\mathbf{A}$ is an $n \times n$ matrix and that k is a (constant) scalar. Show that the set of all vectors $\mathbf{x}$ such that $\mathbf{Ax} = k\mathbf{x}$ is a subspace of $\mathbf{R}^n$.

29. Let $\mathbf{A}$ be an $n \times n$ matrix, $\mathbf{b}$ be a nonzero vector, and $\mathbf{x}_0$ be a solution vector of the system $\mathbf{Ax} = \mathbf{b}$. Show that $\mathbf{x}$ is a solution of the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ is a solution of the homogeneous system $\mathbf{Ay} = \mathbf{0}$.

30. Let U and V be subspaces of the vector space W. Their **intersection** $U \cap V$ is the set of all vectors that are both in U and in V. Show that $U \cap V$ is a subspace of W. If U and V are two planes through the origin in $\mathbf{R}^3$, what is $U \cap V$?

31. Let U and V be subspaces of the vector space W. Their **sum** $U + V$ is the set of all vectors $\mathbf{w}$ of the form

$$\mathbf{w} = \mathbf{u} + \mathbf{v},$$

where $\mathbf{u}$ is in U and $\mathbf{v}$ is in V. Show that $U + V$ is a subspace of W. If U and V are lines through the origin in $\mathbf{R}^3$, what is $U + V$?

4.3 Linear Combinations and Independence of Vectors

In Example 5 of Section 4.2 we solved the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 15x_3 + 7x_4 &= 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 &= 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0. \end{aligned} \tag{1}$$

We found that its solution space W consists of all those vectors $\mathbf{x}$ in $\mathbf{R}^4$ that have the form

$$\mathbf{x} = s(3, 4, 1, 0) + t(2, -3, 0, 1). \tag{2}$$

We therefore can visualize W as the plane in $\mathbf{R}^4$ determined by the vectors $\mathbf{v}_1 = (3, 4, 1, 0)$ and $\mathbf{v}_2 = (2, -3, 0, 1)$. The fact that every solution vector is a combination [as in (2)] of the particular solution vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ gives us a tangible understanding of the solution space W of the system in (1).

More generally, we know from Theorem 2 in Section 4.2 that the solution set V of any $m \times n$ homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is a subspace of $\mathbf{R}^n$. In order to understand such a vector space V better, we would like to find a minimal set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V such that *every* vector in V is a sum of scalar multiples of these particular vectors.

The vector $\mathbf{w}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ provided that there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k. \tag{3}$$

Given a vector $\mathbf{w}$ in $\mathbf{R}^n$, the problem of determining whether or not $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ amounts to solving a linear system to see whether we can find scalars $c_1, c_2, \ldots, c_k$ so that (3) holds.

Example 1 To determine whether the vector $\mathbf{w} = (2, -6, 3)$ in $\mathbf{R}^3$ is a linear combination of the vectors $\mathbf{v}_1 = (1, -2, -1)$ and $\mathbf{v}_2 = (3, -5, 4)$, we write the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$ in matrix form:

$$c_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

THEOREM 3 Independence of Fewer Than n Vectors in $\mathbf{R}^n$

Consider k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$, with $k < n$. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$$

be the $n \times k$ matrix having them as its column vectors. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if and only if some $k \times k$ submatrix of $\mathbf{A}$ has nonzero determinant.

Rather than including a complete proof, we will simply illustrate the "if part of Theorem 3 in the case $n = 5$, $k = 3$. Let $\mathbf{v}_1 = (a_1, a_2, a_3, a_4, a_5)$, $\mathbf{v}_2 =$ $(b_1, b_2, b_3, b_4, b_5)$, and $\mathbf{v}_3 = (c_1, c_2, c_3, c_4, c_5)$ be three vectors in $\mathbf{R}^5$ such that th 5×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{bmatrix}$$

has a 3×3 submatrix with nonzero determinant. Suppose, for instance, that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{vmatrix} \neq 0.$$

Then Theorem 2 implies that the three vectors $\mathbf{u}_1 = (a_1, a_3, a_5)$, $\mathbf{u}_2 = (b_1, b_3, b_5)$ and $\mathbf{u}_3 = (c_1, c_3, c_5)$ in $\mathbf{R}^3$ are linearly independent. Now suppose that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 +$ $c_3\mathbf{v}_3 = \mathbf{0}$. Then by deleting the second and fourth components of each vector in thi equation, we find that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$. But the fact that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ ar linearly independent implies that $c_1 = c_2 = c_3 = 0$, and it now follows that $\mathbf{v}_1$, $\mathbf{v}_2$ $\mathbf{v}_3$ are linearly independent.

4.3 Problems

In Problems 1–8, determine whether the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent or linearly dependent. Do this essentially by inspection—that is, without solving a linear system of equations.

1. $\mathbf{v}_1 = (4, -2, 6, -4)$, $\mathbf{v}_2 = (6, -3, 9, -6)$
2. $\mathbf{v}_1 = (3, 9, -3, 6)$, $\mathbf{v}_2 = (2, 6, -2, 4)$
3. $\mathbf{v}_1 = (3, 4)$, $\mathbf{v}_2 = (6, -1)$, $\mathbf{v}_3 = (7, 5)$
4. $\mathbf{v}_1 = (4, -2, 2)$, $\mathbf{v}_2 = (5, 4, -3)$, $\mathbf{v}_3 = (4, 6, 5)$, $\mathbf{v}_4 = (-7, 9, 3)$
5. $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, -2, 0)$, $\mathbf{v}_3 = (0, 0, 3)$
6. $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (1, 1, 1)$
7. $\mathbf{v}_1 = (2, 1, 0, 0)$, $\mathbf{v}_2 = (3, 0, 1, 0)$, $\mathbf{v}_3 = (4, 0, 0, 1)$
8. $\mathbf{v}_1 = (1, 0, 3, 0)$, $\mathbf{v}_2 = (0, 2, 0, 4)$, $\mathbf{v}_3 = (1, 2, 3, 4)$

In Problems 9–16, express the indicated vector $\mathbf{w}$ as a linear combination of the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if this is possible. If not, show that it is impossible.

9. $\mathbf{w} = (1, 0, -7)$; $\mathbf{v}_1 = (5, 3, 4)$, $\mathbf{v}_2 = (3, 2, 5)$
10. $\mathbf{w} = (3, -1, -2)$; $\mathbf{v}_1 = (-3, 1, -2)$, $\mathbf{v}_2 = (6, -2, 3)$
11. $\mathbf{w} = (1, 0, 0, -1)$; $\mathbf{v}_1 = (7, -6, 4, 5)$, $\mathbf{v}_2 = (3, -3, 2, 3)$
12. $\mathbf{w} = (4, -4, 3, 3)$; $\mathbf{v}_1 = (7, 3, -1, 9)$, $\mathbf{v}_2 = (-2, -2, 1, -3)$
13. $\mathbf{w} = (5, 2, -2)$; $\mathbf{v}_1 = (1, 5, -3)$, $\mathbf{v}_2 = (5, -3, 4)$
14. $\mathbf{w} = (2, -3, 2, -3)$; $\mathbf{v}_1 = (1, 0, 0, 3)$, $\mathbf{v}_2 = (0, 1, -2, 0)$ $\mathbf{v}_3 = (0, -1, 1, 1)$
15. $\mathbf{w} = (4, 5, 6)$; $\mathbf{v}_1 = (2, -1, 4)$, $\mathbf{v}_2 = (3, 0, 1)$, $\mathbf{v}_3 = (1, 2, -1)$
16. $\mathbf{w} = (7, 7, 9, 11)$; $\mathbf{v}_1 = (2, 0, 3, 1)$, $\mathbf{v}_2 = (4, 1, 3, 2)$, $\mathbf{v}_3 = (1, 3, -1, 3)$

In Problems 17–22, three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

17. $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, -3, 4)$, $\mathbf{v}_3 = (3, 5, 2)$
18. $\mathbf{v}_1 = (2, 0, -3)$, $\mathbf{v}_2 = (4, -5, -6)$, $\mathbf{v}_3 = (-2, 1, 3)$
19. $\mathbf{v}_1 = (2, 0, 3, 0)$, $\mathbf{v}_2 = (5, 4, -2, 1)$, $\mathbf{v}_3 = (2, -1, 1, -1)$
20. $\mathbf{v}_1 = (1, 1, -1, 1)$, $\mathbf{v}_2 = (2, 1, 1, 1)$, $\mathbf{v}_3 = (3, 1, 4, 1)$

21. $\mathbf{v}_1 = (3, 0, 1, 2)$, $\mathbf{v}_2 = (1, -1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1, 0)$
22. $\mathbf{v}_1 = (3, 9, 0, 5)$, $\mathbf{v}_2 = (3, 0, 9, -7)$, $\mathbf{v}_3 = (4, 7, 5, 0)$

In Problems 23–26, the vectors $\{\mathbf{v}_i\}$ are known to be linearly independent. Apply the definition of linear independence to show that the vectors $\{\mathbf{u}_i\}$ are also linearly independent.

23. $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2$
24. $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_2$
25. $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + 2\mathbf{v}_2$, $\mathbf{u}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$
26. $\mathbf{u}_1 = \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2$
27. *Prove*: If the (finite) set S of vectors contains the zero vector, then S is linearly dependent.
28. *Prove*: If the set S of vectors is linearly dependent and the (finite) set T contains S, then T is also linearly dependent. You may assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and that $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ with $m > k$.
29. Show that if the (finite) set S of vectors is linearly independent, then any subset T of S is also linearly independent.
30. Suppose that the subspace U of the vector space V contains the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Show that U contains the subspace spanned by these vectors.
31. Let S and T be sets of vectors in a vector space such that S is a subset of span(T). Show that span(S) is also a subset of span(T).
32. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be linearly independent vectors in the set S of vectors. *Prove*: If no set of more than k vectors in S is linearly independent, then every vector in S is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

In Problems 33–35, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbf{R}^n$ and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$$

be the $n \times k$ matrix with these vectors as its column vectors.

33. *Prove*: If some $k \times k$ submatrix of $\mathbf{A}$ is the $k \times k$ identity matrix, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.
34. Suppose that $k = n$, that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, and that $\mathbf{B}$ is a nonsingular $n \times n$ matrix. Prove that the column vectors of the matrix $\mathbf{AB}$ are linearly independent.
35. Suppose that $k < n$, that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, and that $\mathbf{B}$ is a nonsingular $k \times k$ matrix. Use Theorem 3 to show that the column vectors of $\mathbf{AB}$ are linearly independent.

4.4 Bases and Dimension for Vector Spaces

An especially useful way of describing the solution space of a homogeneous linear system is to list explicitly a set S of solution vectors such that *every* solution vector is a unique linear combination of these particular ones. The following definition specifies the properties of such a set S of "basic" solution vectors, and the concept is equally important for vector spaces other than solution spaces.

DEFINITION Basis

A finite set S of vectors in a vector space V is called a **basis** for V provided that
(a) the vectors in S are linearly independent, and
(b) the vectors in S span V.

In short, a basis for the vector space V is a linearly independent spanning set of vectors in V. Thus, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then any vector $\mathbf{w}$ in V can be expressed as a linear combination

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{1}$$

of the vectors in S, and we saw in Section 4.3 that the linear independence of S implies that the coefficients $c_1, c_2, \ldots, c_n$ in (1) are *unique*. That is, $\mathbf{w}$ cannot be expressed differently as a linear combination of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Example 1 The **standard basis** for $\mathbf{R}^n$ consists of the unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector in $\mathbf{R}^n$, then

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

With $t = 1$ and $r = s = 0$,

we obtain $\quad \mathbf{v}_3 = (-3, 0, -4, 0, 1)$.

Thus the solution space of the system in (14) is a 3-dimensional subspace of $\mathbf{R}^5$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. ■

4.4 Problems

In Problems 1–8, determine whether or not the given vectors in $\mathbf{R}^n$ form a basis for $\mathbf{R}^n$.

1. $\mathbf{v}_1 = (4, 7)$, $\mathbf{v}_2 = (5, 6)$
2. $\mathbf{v}_1 = (3, -1, 2)$, $\mathbf{v}_2 = (6, -2, 4)$, $\mathbf{v}_3 = (5, 3, -1)$
3. $\mathbf{v}_1 = (1, 7, -3)$, $\mathbf{v}_2 = (2, 1, 4)$, $\mathbf{v}_3 = (6, 5, 1)$, $\mathbf{v}_4 = (0, 7, 13)$
4. $\mathbf{v}_1 = (3, -7, 5, 2)$, $\mathbf{v}_2 = (1, -1, 3, 4)$, $\mathbf{v}_3 = (7, 11, 3, 13)$
5. $\mathbf{v}_1 = (0, 7, -3)$, $\mathbf{v}_2 = (0, 5, 4)$, $\mathbf{v}_3 = (0, 5, 10)$
6. $\mathbf{v}_1 = (0, 0, 1)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (1, 2, 3)$
7. $\mathbf{v}_1 = (0, 0, 1)$, $\mathbf{v}_2 = (7, 4, 11)$, $\mathbf{v}_3 = (5, 3, 13)$
8. $\mathbf{v}_1 = (2, 0, 0, 0)$, $\mathbf{v}_2 = (0, 3, 0, 0)$, $\mathbf{v}_3 = (0, 0, 7, 6)$, $\mathbf{v}_4 = (0, 0, 4, 5)$

In Problems 9–11, find a basis for the indicated subspace of $\mathbf{R}^3$.

9. The plane with equation $x - 2y + 5z = 0$.
10. The plane with equation $y = z$.
11. The line of intersection of the planes described in Problems 9 and 10.

In Problems 12–14, find a basis for the indicated subspace of $\mathbf{R}^4$.

12. The set of all vectors of the form (a, b, c, d) for which $a = b + c + d$.
13. The set of all vectors of the form (a, b, c, d) such that $a = 3c$ and $b = 4d$.
14. The set of all vectors of the form (a, b, c, d) for which $a + 2b = c + 3d = 0$.

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system.

15. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 - 3x_2 - x_3 &= 0 \end{aligned}$
16. $\begin{aligned} x_1 + 3x_2 + 4x_3 &= 0 \\ 3x_1 + 8x_2 + 7x_3 &= 0 \end{aligned}$
17. $\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= 0 \\ 2x_1 - 5x_2 + 7x_3 - 3x_4 &= 0 \end{aligned}$
18. $\begin{aligned} x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 0 \end{aligned}$
19. $\begin{aligned} x_1 - 3x_2 - 9x_3 - 5x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 + 11x_4 &= 0 \\ x_1 + 3x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$
20. $\begin{aligned} x_1 - 3x_2 - 10x_3 + 5x_4 &= 0 \\ x_1 + 4x_2 + 11x_3 - 2x_4 &= 0 \\ x_1 + 3x_2 + 8x_3 - x_4 &= 0 \end{aligned}$
21. $\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$
22. $\begin{aligned} x_1 - 2x_2 - 3x_3 - 16x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + 17x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 + 26x_4 &= 0 \end{aligned}$
23. $\begin{aligned} x_1 + 5x_2 + 13x_3 + 14x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \\ 2x_1 + 7x_2 + 17x_3 + 19x_4 &= 0 \end{aligned}$
24. $\begin{aligned} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 &= 0 \\ x_1 \phantom{{}+3x_2} + 2x_3 + x_4 + 3x_5 &= 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 &= 0 \end{aligned}$
25. $\begin{aligned} x_1 + 2x_2 + 7x_3 - 9x_4 + 31x_5 &= 0 \\ 2x_1 + 4x_2 + 7x_3 - 11x_4 + 34x_5 &= 0 \\ 3x_1 + 6x_2 + 5x_3 - 11x_4 + 29x_5 &= 0 \end{aligned}$
26. $\begin{aligned} 3x_1 + x_2 - 3x_3 + 11x_4 + 10x_5 &= 0 \\ 5x_1 + 8x_2 + 2x_3 - 2x_4 + 7x_5 &= 0 \\ 2x_1 + 5x_2 \phantom{{}+2x_3} - x_4 + 14x_5 &= 0 \end{aligned}$

Problems 27 through 36 further explore independent sets, spanning sets, and bases.

27. Suppose that S is a set of n linearly independent vectors in the n-dimensional vector space V. Prove that S is a basis for V.
28. Suppose that S is a set of n vectors that span the n-dimensional vector space V. Prove that S is a basis for V.
29. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for the proper subspace W of the vector space V, and suppose that the vector $\mathbf{v}$ of V is not in W. Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}$ are linearly independent.
30. Use the result of Problem 29 to prove that every linearly independent set of vectors in a finite-dimensional vector space V is contained in a basis for V.
31. Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ span the vector space V and that $\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ span V.
32. Use the result of Problem 31 to prove that every spanning set for a finite-dimensional vector space V contains a basis for V.
33. Let S be a linearly independent set of vectors in the finite-dimensional vector space V. Then S is called a **maximal linearly independent set** provided that if any other vector is adjoined to S, then the resulting set is linearly dependent. Prove that every maximal linearly independent set in V is a basis for V.

34. Let S be a finite set of vectors that span the vector space V. Then S is called a **minimal spanning set** provided that no proper subset of S spans V. Prove that every minimal spanning set in V is a basis for V.

35. Let S be a finite set of vectors that span the vector space V. Then S is called a **uniquely spanning set** provided that each vector in V can be expressed in one and only one way as a linear combination of the vectors in S. Prove that every uniquely spanning set in V is a basis for V.

36. Apply the definition of linear independence to show directly that the column vectors of the matrix in (12) are linearly independent.

4.5 Row and Column Spaces

In numerous examples we have observed the phenomenon of "disappearing equa- tions" that sometimes occurs when we solve a linear system using the method o Gaussian elimination. The appearance in this process of a trivial equation 0 = (means that one of the original equations was redundant. For instance, in the systen

$$\begin{aligned} x - 2y + 2z &= 0 \\ x + 4y + 3z &= 0 \\ 2x + 2y + 5z &= 0, \end{aligned}$$

the third equation provides no additional information about a solution (x, y, z) be cause it is merely the sum of the first two equations.

Given a homogeneous linear system, it is natural to ask how many of the equa tions are "irredundant," and which ones they are. We will see that an answer to th question leads to a natural and simple relation between the number of irredunda equations, the number of unknowns, and the number of linearly independent solu tions.

Row Space and Row Rank

The individual equations of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ correspond the "row matrices"

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

of the $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$. The **row vectors** of $\mathbf{A}$ are the m vectors

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{r}_2 &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{r}_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned}$$

in $\mathbf{R}^n$. Recalling from Section 3.4 the convection that n-tuples denote colun vector elements of $\mathbf{R}^n$, we see that the row vectors of $\mathbf{A}$ are the transposes of its r matrices; that is,

$$\mathbf{r}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}^T$$