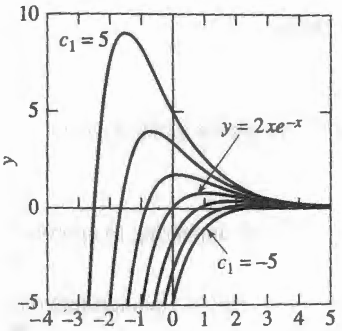

FIGURE 5.1.10. Solutions $y(x) = c_1e^{-x} + 2xe^{-x}$ of $y'' + 2y' + y = 0$ with different values of c_1.

Differentiation yields

$$y'(x) = -c_1e^{-x} + c_2e^{-x} - c_2xe^{-x},$$

so the initial conditions yield the equations

$$\begin{aligned} y(0) &= c_1 &= 5, \\ y'(0) &= -c_1 + c_2 &= -3, \end{aligned}$$

which imply that $c_1 = 5$ and $c_2 = 2$. Thus the desired particular solution of the initial value problem is

$$y(x) = 5e^{-x} + 2xe^{-x}.$$

This particular solution, together with several others of the form $y(x) = c_1e^{-x} + 2xe^{-x}$, is illustrated in Fig. 5.1.10. ∎

The characteristic equation in (18) may have either real or complex roots. The case of complex roots will be discussed in Section 5.3.

5.1 Problems

In Problems 1 through 16, a homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

1. $y'' - y = 0$; $y_1 = e^x$, $y_2 = e^{-x}$; $y(0) = 0$, $y'(0) = 5$
2. $y'' - 9y = 0$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$; $y(0) = -1$, $y'(0) = 15$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$; $y(0) = 3$, $y'(0) = 8$
4. $y'' + 25y = 0$; $y_1 = \cos 5x$, $y_2 = \sin 5x$; $y(0) = 10$, $y'(0) = -10$
5. $y'' - 3y' + 2y = 0$; $y_1 = e^x$, $y_2 = e^{2x}$; $y(0) = 1$, $y'(0) = 0$
6. $y'' + y' - 6y = 0$; $y_1 = e^{2x}$, $y_2 = e^{-3x}$; $y(0) = 7$, $y'(0) = -1$
7. $y'' + y' = 0$; $y_1 = 1$, $y_2 = e^{-x}$; $y(0) = -2$, $y'(0) = 8$
8. $y'' - 3y' = 0$; $y_1 = 1$, $y_2 = e^{3x}$; $y(0) = 4$, $y'(0) = -2$
9. $y'' + 2y' + y = 0$; $y_1 = e^{-x}$, $y_2 = xe^{-x}$; $y(0) = 2$, $y'(0) = -1$
10. $y'' - 10y' + 25y = 0$; $y_1 = e^{5x}$, $y_2 = xe^{5x}$; $y(0) = 3$, $y'(0) = 13$
11. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$; $y(0) = 0$, $y'(0) = 5$
12. $y'' + 6y' + 13y = 0$; $y_1 = e^{-3x}\cos 2x$, $y_2 = e^{-3x}\sin 2x$; $y(0) = 2$, $y'(0) = 0$
13. $x^2y'' - 2xy' + 2y = 0$; $y_1 = x$, $y_2 = x^2$; $y(1) = 3$, $y'(1) = 1$
14. $x^2y'' + 2xy' - 6y = 0$; $y_1 = x^2$, $y_2 = x^{-3}$; $y(2) = 10$, $y'(2) = 15$
15. $x^2y'' - xy' + y = 0$; $y_1 = x$, $y_2 = x \ln x$; $y(1) = 7$, $y'(1) = 2$
16. $x^2y'' + xy' + y = 0$; $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$; $y(1) = 2$, $y'(1) = 3$

The following three problems illustrate the fact that the superposition principle does not generally hold for nonlinear equations.

17. Show that $y = 1/x$ is a solution of $y' + y^2 = 0$, but that if $c \neq 0$ and $c \neq 1$, then $y = c/x$ is not a solution.
18. Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.
19. Show that $y_1 \equiv 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

Determine whether the pairs of functions in Problems 20 through 26 are linearly independent or linearly dependent on the real line.

20. $f(x) = \pi$, $g(x) = \cos^2 x + \sin^2 x$
21. $f(x) = x^3$, $g(x) = x^2|x|$
22. $f(x) = 1 + x$, $g(x) = 1 + |x|$
23. $f(x) = xe^x$, $g(x) = |x|e^x$
24. $f(x) = \sin^2 x$, $g(x) = 1 - \cos 2x$
25. $f(x) = e^x \sin x$, $g(x) = e^x \cos x$
26. $f(x) = 2\cos x + 3\sin x$, $g(x) = 3\cos x - 2\sin x$
27. Let y_p be a particular solution of the nonhomogeneous equation $y'' + py' + qy = f(x)$ and let y_c be a solution of its associated homogeneous equation. Show that $y = y_c + y_p$ is a solution of the given nonhomogeneous equation.
28. With $y_p = 1$ and $y_c = c_1 \cos x + c_2 \sin x$ in the notation of Problem 27, find a solution of $y'' + y = 1$ satisfying the initial conditions $y(0) = -1 = y'(0)$.

Problems 29 through 32 explore the propeties of the Wronskian.

29. Show that $y_1 = x^2$ and $y_2 = x^3$ are two different solutions of $x^2y'' - 4xy' + 6y = 0$, both satisfying the initial conditions $y(0) = 0 = y'(0)$. Explain why these facts do not contradict Theorem 2 (with respect to the guaranteed uniqueness).
30. **(a)** Show that $y_1 = x^3$ and $y_2 = |x^3|$ are linearly independent solutions on the real line of the equation $x^2y'' - 3xy' + 3y = 0$. **(b)** Verify that $W(y_1, y_2)$ is identically zero. Why do these facts not contradict Theorem 3?

31. Show that $y_1 = \sin x^2$ and $y_2 = \cos x^2$ are linearly independent functions, but that their Wronskian vanishes at $x = 0$. Why does this imply that there is *no* differential equation of the form $y'' + p(x)y' + q(x)y = 0$, with both p and q continuous everywhere, having both y_1 and y_2 as solutions?

32. Let y_1 and y_2 be two solutions of $A(x)y'' + B(x)y' + C(x)y = 0$ on an open interval I where A, B, and C are continuous and $A(x)$ is never zero. **(a)** Let $W = W(y_1, y_2)$. Show that

$$A(x)\frac{dW}{dx} = (y_1)(Ay_2'') - (y_2)(Ay_1'').$$

Then substitute for Ay_2'' and Ay_1'' from the original differential equation to show that

$$A(x)\frac{dW}{dx} = -B(x)W(x).$$

(b) Solve this first-order equation to deduce **Abel's formula**

$$W(x) = K\exp\left(-\int \frac{B(x)}{A(x)}\,dx\right),$$

where K is a constant. **(c)** Why does Abel's formula imply that the Wronskian $W(y_1, y_2)$ is either zero everywhere or nonzero everywhere (as stated in Theorem 3)?

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

33. $y'' - 3y' + 2y = 0$
34. $y'' + 2y' - 15y = 0$
35. $y'' + 5y' = 0$
36. $2y'' + 3y' = 0$
37. $2y'' - y' - y = 0$
38. $4y'' + 8y' + 3y = 0$
39. $4y'' + 4y' + y = 0$
40. $9y'' - 12y' + 4y = 0$
41. $6y'' - 7y' - 20y = 0$
42. $35y'' - y' - 12y = 0$

Each of Problems 43 through 48 gives a general solution $y(x)$ of a homogeneous second-order differential equation $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation.

43. $y(x) = c_1 + c_2e^{-10x}$
44. $y(x) = c_1e^{10x} + c_2e^{-10x}$
45. $y(x) = c_1e^{-10x} + c_2xe^{-10x}$
46. $y(x) = c_1e^{10x} + c_2e^{100x}$
47. $y(x) = c_1 + c_2x$
48. $y(x) = e^x\left(c_1e^{x\sqrt{2}} + c_2e^{-x\sqrt{2}}\right)$

Problems 49 and 50 deal with the solution curves of $y'' + 3y' + 2y = 0$ shown in Figs. 5.1.6 and 5.1.7.

49. Find the highest point on the solution curve with $y(0) = 1$ and $y'(0) = 6$ in Fig. 5.1.6.

50. Figure 5.1.7 suggests that the solution curves shown all meet at a common point in the third quadrant. Assuming that this is indeed the case, find the coordinates of that point.

51. A second-order **Euler equation** is one of the form

$$ax^2y'' + bxy' + cy = 0 \tag{22}$$

where a, b, c are constants. **(a)** Show that if $x > 0$, then the substitution $v = \ln x$ transforms Eq. (22) into the constant-coefficient linear equation

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0 \tag{23}$$

with independent variable v. **(b)** If the roots r_1 and r_2 of the characteristic equation of Eq. (23) are real and distinct, conclude that a general solution of the Euler equation in (22) is $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

Make the substitution $v = \ln x$ of Problem 51 to find general solutions (for $x > 0$) of the Euler equations in Problems 52–56.

52. $x^2y'' + xy' - y = 0$
53. $x^2y'' + 2xy' - 12y = 0$
54. $4x^2y'' + 8xy' - 3y = 0$
55. $x^2y'' + xy' = 0$
56. $x^2y'' - 3xy' + 4y = 0$

Application Plotting Second-Order Solution Families

Go to goo.gl/jRVUO1 to download this application's computing resources including Maple/Mathematica/MATLAB.

This application deals with the plotting by computer of families of solutions such as those illustrated in Figs. 5.1.6 and 5.1.7. Show first that the general solution of the differential equation

$$y'' + 3y' + 2y = 0 \tag{1}$$

is

$$y(x) = c_1e^{-x} + c_2e^{-2x}. \tag{2}$$

This is equivalent to the graphing calculator result shown in Figure 5.1.11, and to the Wolfram|Alpha output generated by the simple query

```
y'' + 3y' + 2y = 0
```

Solution The general solution of Eq. (17) is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + 3x.$$

Now

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x + 3.$$

Hence the initial conditions give

$$y(0) = c_1 = 5,$$
$$y'(0) = 2c_2 + 3 = 7.$$

We find that $c_1 = 5$ and $c_2 = 2$. Thus the desired solution is

$$y(x) = 5\cos 2x + 2\sin 2x + 3x.$$

5.2 Problems

In Problems 1 through 6, show directly that the given functions are linearly dependent on the real line. That is, find a nontrivial linear combination of the given functions that vanishes identically.

1. $f(x) = 2x$, $g(x) = 3x^2$, $h(x) = 5x - 8x^2$
2. $f(x) = 5$, $g(x) = 2 - 3x^2$, $h(x) = 10 + 15x^2$
3. $f(x) = 0$, $g(x) = \sin x$, $h(x) = e^x$
4. $f(x) = 17$, $g(x) = 2\sin^2 x$, $h(x) = 3\cos^2 x$
5. $f(x) = 17$, $g(x) = \cos^2 x$, $h(x) = \cos 2x$
6. $f(x) = e^x$, $g(x) = \cosh x$, $h(x) = \sinh x$

In Problems 7 through 12, use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

7. $f(x) = 1$, $g(x) = x$, $h(x) = x^2$; the real line
8. $f(x) = e^x$, $g(x) = e^{2x}$, $h(x) = e^{3x}$; the real line
9. $f(x) = e^x$, $g(x) = \cos x$, $h(x) = \sin x$; the real line
10. $f(x) = e^x$, $g(x) = x^{-2}$, $h(x) = x^{-2}\ln x$; $x > 0$
11. $f(x) = x$, $g(x) = xe^x$, $h(x) = x^2e^x$; the real line
12. $f(x) = x$, $g(x) = \cos(\ln x)$, $h(x) = \sin(\ln x)$; $x > 0$

In Problems 13 through 20, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

13. $y^{(3)} + 2y'' - y' - 2y = 0$; $y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$; $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{-2x}$
14. $y^{(3)} - 6y'' + 11y' - 6y = 0$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 3$; $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$
15. $y^{(3)} - 3y'' + 3y' - y = 0$; $y(0) = 2$, $y'(0) = 0$, $y''(0) = 0$; $y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$
16. $y^{(3)} - 5y'' + 8y' - 4y = 0$; $y(0) = 1$, $y'(0) = 4$, $y''(0) = 0$; $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = xe^{2x}$
17. $y^{(3)} + 9y' = 0$; $y(0) = 3$, $y'(0) = -1$, $y''(0) = 2$; $y_1 = 1$, $y_2 = \cos 3x$, $y_3 = \sin 3x$
18. $y^{(3)} - 3y'' + 4y' - 2y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$; $y_1 = e^x$, $y_2 = e^x\cos x$, $y_3 = e^x \sin x$.
19. $x^3y^{(3)} - 3x^2y'' + 6xy' - 6y = 0$; $y(1) = 6$, $y'(1) = 14$, $y''(1) = 22$; $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$
20. $x^3y^{(3)} + 6x^2y'' + 4xy' - 4y = 0$; $y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$; $y_1 = x$, $y_2 = x^{-2}$, $y_3 = x^{-2}\ln x$

In Problems 21 through 24, a nonhomogeneous differential equation, a complementary solution y_c, and a particular solution y_p are given. Find a solution satisfying the given initial conditions.

21. $y'' + y = 3x$; $y(0) = 2$, $y'(0) = -2$; $y_c = c_1\cos x + c_2 \sin x$; $y_p = 3x$
22. $y'' - 4y = 12$; $y(0) = 0$, $y'(0) = 10$; $y_c = c_1e^{2x} + c_2e^{-2x}$; $y_p = -3$
23. $y'' - 2y' - 3y = 6$; $y(0) = 3$, $y'(0) = 11$; $y_c = c_1e^{-x} + c_2e^{3x}$; $y_p = -2$
24. $y'' - 2y' + 2y = 2x$; $y(0) = 4$, $y'(0) = 8$; $y_c = c_1e^x\cos x + c_2e^x \sin x$; $y_p = x + 1$
25. Let $Ly = y'' + py' + qy$. Suppose that y_1 and y_2 are two functions such that
$$Ly_1 = f(x) \quad \text{and} \quad Ly_2 = g(x).$$
Show that their sum $y = y_1 + y_2$ satisfies the nonhomogeneous equation $Ly = f(x) + g(x)$.
26. **(a)** Find by inspection particular solutions of the two nonhomogeneous equations
$$y'' + 2y = 4 \quad \text{and} \quad y'' + 2y = 6x.$$
(b) Use the method of Problem 25 to find a particular solution of the differential equation $y'' + 2y = 6x + 4$.
27. Prove directly that the functions
$$f_1(x) \equiv 1, \quad f_2(x) = x, \quad \text{and} \quad f_3(x) = x^2$$
are linearly independent on the whole real line. (*Suggestion:* Assume that $c_1 + c_2x + c_3x^2 = 0$. Differentiate this equation twice, and conclude from the equations you get that $c_1 = c_2 = c_3 = 0$.)

28. Generalize the method of Problem 27 to prove directly that the functions

$$f_0(x) \equiv 1, \quad f_1(x) = x, \quad f_2(x) = x^2, \ldots, \quad f_n(x) = x^n$$

are linearly independent on the real line.

29. Use the result of Problem 28 and the definition of linear independence to prove directly that, for any constant r, the functions

$$f_0(x) = e^{rx}, \quad f_1(x) = xe^{rx}, \quad \ldots, \quad f_n(x) = x^n e^{rx}$$

are linearly independent on the whole real line.

30. Verify that $y_1 = x$ and $y_2 = x^2$ are linearly independent solutions on the entire real line of the equation

$$x^2 y'' - 2xy' + 2y = 0,$$

but that $W(x, x^2)$ vanishes at $x = 0$. Why do these observations not contradict part (b) of Theorem 3?

31. This problem indicates why we can impose *only* n initial conditions on a solution of an nth-order linear differential equation. **(a)** Given the equation

$$y'' + py' + qy = 0,$$

explain why the value of $y''(a)$ is determined by the values of $y(a)$ and $y'(a)$. **(b)** Prove that the equation

$$y'' - 2y' - 5y = 0$$

has a solution satisfying the conditions

$$y(0) = 1, \quad y'(0) = 0, \quad \text{and} \quad y''(0) = C$$

if and only if $C = 5$.

32. Prove that an nth-order homogeneous linear differential equation satisfying the hypotheses of Theorem 2 has n linearly independent solutions $y_1, y_2, \ldots, y_n$. (*Suggestion:* Let y_i be the unique solution such that

$$y_i^{(i-1)}(a) = 1 \quad \text{and} \quad y_i^{(k)}(a) = 0 \quad \text{if } k \neq i - 1.)$$

33. Suppose that the three numbers r_1, r_2, and r_3 are distinct. Show that the three functions $\exp(r_1 x)$, $\exp(r_2 x)$, and $\exp(r_3 x)$ are linearly independent by showing that their Wronskian

$$W = \exp[(r_1 + r_2 + r_3)x] \cdot \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix}$$

is nonzero for all x.

34. Assume as known that the **Vandermonde determinant**

$$V = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

is nonzero if the numbers $r_1, r_2, \ldots, r_n$ are distinct. Prove by the method of Problem 33 that the functions

$$f_i(x) = \exp(r_i x), \quad 1 \leqq i \leqq n$$

are linearly independent.

35. According to Problem 32 of Section 5.1, the Wronskian $W(y_1, y_2)$ of two solutions of the second-order equation

$$y'' + p_1(x)y' + p_2(x)y = 0$$

is given by Abel's formula

$$W(x) = K \exp\left(-\int p_1(x)\,dx\right)$$

for some constant K. It can be shown that the Wronskian of n solutions $y_1, y_2, \ldots, y_n$ of the nth-order equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

satisfies the same identity. Prove this for the case $n = 3$ as follows: **(a)** The derivative of a determinant of functions is the sum of the determinants obtained by separately differentiating the rows of the original determinant. Conclude that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}.$$

(b) Substitute for $y_1^{(3)}$, $y_2^{(3)}$, and $y_3^{(3)}$ from the equation

$$y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0,$$

and then show that $W' = -p_1 W$. Integration now gives Abel's formula.

36. Suppose that one solution $y_1(x)$ of the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{18}$$

is known (on an interval I where p and q are continuous functions). The method of **reduction of order** consists of substituting $y_2(x) = v(x)y_1(x)$ in (18) and attempting to determine the function $v(x)$ so that $y_2(x)$ is a second linearly independent solution of (18). After substituting $y = v(x)y_1(x)$ in Eq. (18), use the fact that $y_1(x)$ is a solution to deduce that

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{19}$$

If $y_1(x)$ is known, then (19) is a separable equation that is readily solved for the derivative $v'(x)$ of $v(x)$. Integration of $v'(x)$ then gives the desired (nonconstant) function $v(x)$.

37. Before applying Eq. (19) with a given homogeneous second-order linear differential equation and a known solution $y_1(x)$, the equation must first be written in the form of (18) with leading coefficient 1 in order to correctly determine the coefficient function $p(x)$. Frequently it is more convenient to simply substitute $y = v(x)y_1(x)$ in the given differential equation and then proceed directly to find $v(x)$. Thus, starting with the readily verified solution $y_1(x) = x^3$ of the equation

$$x^2y'' - 5xy' + 9y = 0 \quad (x > 0),$$

substitute $y = vx^3$ and deduce that $xv'' + v' = 0$. Thence solve for $v(x) = C \ln x$, and thereby obtain (with $C = 1$) the second solution $y_2(x) = x^3 \ln x$.

In each of Problems 38 through 42, a differential equation and one solution y_1 are given. Use the method of reduction of order as in Problem 37 to find a second linearly independent solution y_2.

38. $x^2y'' + xy' - 9y = 0$ $(x > 0)$; $y_1(x) = x^3$

39. $4y'' - 4y' + y = 0$; $y_1(x) = e^{x/2}$

40. $x^2y'' - x(x+2)y' + (x+2)y = 0$ $(x > 0)$; $y_1(x) = x$

41. $(x+1)y'' - (x+2)y' + y = 0$ $(x > -1)$; $y_1(x) = e^x$

42. $(1-x^2)y'' + 2xy' - 2y = 0$ $(-1 < x < 1)$; $y_1(x) = x$

43. First note that $y_1(x) = x$ is one solution of Legendre's equation of order 1,

$$(1-x^2)y'' - 2xy' + 2y = 0.$$

Then use the method of reduction of order to derive the second solution

$$y_2(x) = 1 - \frac{x}{2}\ln\frac{1+x}{1-x} \quad (\text{for } -1 < x < 1).$$

44. First verify by substitution that $y_1(x) = x^{-1/2}\cos x$ is one solution (for $x > 0$) of Bessel's equation of order $\frac{1}{2}$,

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0.$$

Then derive by reduction of order the second solution $y_2(x) = x^{-1/2}\sin x$.

5.2 Application Plotting Third-Order Solution Families

Go to goo.gl/rj9zB2 to download this application's computing resources including *Maple*/*Mathematica*/MATLAB.

This application deals with the plotting by computer of families of solutions such as those illustrated in Figs. 5.2.2 through 5.2.4. We know from Example 6 that the general solution of

$$y^{(3)} + 3y'' + 4y' + 12y = 0 \tag{1}$$

is

$$y(x) = c_1e^{-3x} + c_2\cos 2x + c_3\sin 2x. \tag{2}$$

For Fig. 5.2.2, use the method of Example 6 to show that the particular solution of Eq. (1) satisfying the initial conditions $y(0) = a$, $y'(0) = 0$, and $y''(0) = 0$ is given by

$$y(x) = \frac{a}{13}\left(4e^{-3x} + 9\cos 2x + 6\sin 2x\right). \tag{3}$$

The MATLAB loop

```
x = -1.5 :  0.02 :  5 % x-vector from x = -1.5 to x = 5
for a = -3 :  1 :  3  % for a = -3 to 3 with da = 1 do
    c1 = 4*a/13;
    c2 = 9*a/13;
    c3 = 6*a/13;
    y = c1*exp(-3*x) + c2*cos(2*x) + c3*sin(2*x);
    plot(x,y)
end
```

was used to generate Fig. 5.2.2.

5.3 Problems

Find the general solutions of the differential equations in Problems 1 through 20.

1. $y'' - 4y = 0$
2. $2y'' - 3y' = 0$
3. $y'' + 3y' - 10y = 0$
4. $2y'' - 7y' + 3y = 0$
5. $y'' + 6y' + 9y = 0$
6. $y'' + 5y' + 5y = 0$
7. $4y'' - 12y' + 9y = 0$
8. $y'' - 6y' + 13y = 0$
9. $y'' + 8y' + 25y = 0$
10. $5y^{(4)} + 3y^{(3)} = 0$
11. $y^{(4)} - 8y^{(3)} + 16y'' = 0$
12. $y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$
13. $9y^{(3)} + 12y'' + 4y' = 0$
14. $y^{(4)} + 3y'' - 4y = 0$
15. $y^{(4)} - 8y'' + 16y = 0$
16. $y^{(4)} + 18y'' + 81y = 0$
17. $6y^{(4)} + 11y'' + 4y = 0$
18. $y^{(4)} = 16y$
19. $y^{(3)} + y'' - y' - y = 0$
20. $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$ (*Suggestion:* Expand $(r^2 + r + 1)^2$.)

Solve the initial value problems given in Problems 21 through 26.

21. $y'' - 4y' + 3y = 0$; $y(0) = 7$, $y'(0) = 11$
22. $9y'' + 6y' + 4y = 0$; $y(0) = 3$, $y'(0) = 4$
23. $y'' - 6y' + 25y = 0$; $y(0) = 3$, $y'(0) = 1$
24. $2y^{(3)} - 3y'' - 2y' = 0$; $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$
25. $3y^{(3)} + 2y'' = 0$; $y(0) = -1$, $y'(0) = 0$, $y''(0) = 1$
26. $y^{(3)} + 10y'' + 25y' = 0$; $y(0) = 3$, $y'(0) = 4$, $y''(0) = 5$

Find general solutions of the equations in Problems 27 through 32. First find a small integral root of the characteristic equation by inspection; then factor by division.

27. $y^{(3)} + 3y'' - 4y = 0$
28. $2y^{(3)} - y'' - 5y' - 2y = 0$
29. $y^{(3)} + 27y = 0$
30. $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$
31. $y^{(3)} + 3y'' + 4y' - 8y = 0$
32. $y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

In Problems 33 through 36, one solution of the differential equation is given. Find the general solution.

33. $y^{(3)} + 3y'' - 54y = 0$; $y = e^{3x}$
34. $3y^{(3)} - 2y'' + 12y' - 8y = 0$; $y = e^{2x/3}$
35. $6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0$; $y = \cos 2x$
36. $9y^{(3)} + 11y'' + 4y' - 14y = 0$; $y = e^{-x} \sin x$
37. Find a function $y(x)$ such that $y^{(4)}(x) = y^{(3)}(x)$ for all x and $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, and $y^{(3)}(0) = 7$.
38. Solve the initial value problem

$$y^{(3)} - 5y'' + 100y' - 500y = 0;$$
$$y(0) = 0, \quad y'(0) = 10, \quad y''(0) = 250$$

given that $y_1(x) = e^{5x}$ is one particular solution of the differential equation.

In Problems 39 through 42, find a linear homogeneous constant-coefficient equation with the given general solution.

39. $y(x) = (A + Bx + Cx^2)e^{2x}$
40. $y(x) = Ae^{2x} + B\cos 2x + C\sin 2x$
41. $y(x) = A\cos 2x + B\sin 2x + C\cosh 2x + D\sinh 2x$
42. $y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$

Problems 43 through 47 pertain to the solution of differential equations with complex coefficients.

43. (a) Use Euler's formula to show that every complex number can be written in the form $re^{i\theta}$, where $r \geqq 0$ and $-\pi < \theta \leqq \pi$. (b) Express the numbers 4, -2, $3i$, $1 + i$, and $-1 + i\sqrt{3}$ in the form $re^{i\theta}$. (c) The two square roots of $re^{i\theta}$ are $\pm\sqrt{r}e^{i\theta/2}$. Find the square roots of the numbers $2 - 2i\sqrt{3}$ and $-2 + 2i\sqrt{3}$.
44. Use the quadratic formula to solve the following equations. Note in each case that the roots are not complex conjugates.
    (a) $x^2 + ix + 2 = 0$ (b) $x^2 - 2ix + 3 = 0$
45. Find a general solution of $y'' - 2iy' + 3y = 0$.
46. Find a general solution of $y'' - iy' + 6y = 0$.
47. Find a general solution of $y'' = (-2 + 2i\sqrt{3})y$.
48. Solve the initial value problem

$$y^{(3)} = y; \quad y(0) = 1, \quad y'(0) = y''(0) = 0.$$

(*Suggestion:* Impose the given initial conditions on the general solution

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x},$$

where α and β are the complex conjugate roots of $r^3 - 1 = 0$, to discover that

$$y(x) = \frac{1}{3}\left(e^x + 2e^{-x/2}\cos\frac{x\sqrt{3}}{2}\right)$$

is a solution.)

49. Solve the initial value problem

$$y^{(4)} = y^{(3)} + y'' + y' + 2y;$$
$$y(0) = y'(0) = y''(0) = 0, 2y^{(3)}(0) = 30.$$

50. The differential equation

$$y'' + (\operatorname{sgn} x)y = 0 \qquad (25)$$

has the discontinuous coefficient function

$$\operatorname{sgn} x = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that Eq. (25) nevertheless has two linearly independent solutions $y_1(x)$ and $y_2(x)$ defined for all x such that

- Each satisfies Eq. (25) at each point $x \neq 0$;
- Each has a continuous derivative at $x = 0$;
- $y_1(0) = y_2'(0) = 1$ and $y_2(0) = y_1'(0) = 0$.

(*Suggestion:* Each $y_i(x)$ will be defined by one formula for $x < 0$ and by another for $x \geqq 0$.) The graphs of these two solutions are shown in Fig. 5.3.2.

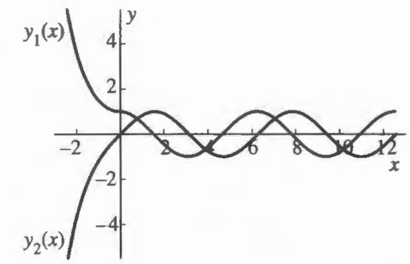

FIGURE 5.3.2. Graphs of $y_1(x)$ and $y_2(x)$ in Problem 50.

51. According to Problem 51 in Section 5.1, the substitution $v = \ln x$ $(x > 0)$ transforms the second-order Euler equation $ax^2y'' + bxy' + cy = 0$ to a constant-coefficient homogeneous linear equation. Show similarly that this same substitution transforms the third-order Euler equation

$$ax^3y''' + bx^2y'' + cxy' + dy = 0$$

(where a, b, c, d are constants) into the constant-coefficient equation

$$a\frac{d^3y}{dv^3} + (b - 3a)\frac{d^2y}{dv^2} + (c - b + 2a)\frac{dy}{dv} + dy = 0.$$

Make the substitution $v = \ln x$ of Problem 51 to find general solutions (for $x > 0$) of the Euler equations in Problems 52 through 58.

52. $x^2y'' + xy' + 9y = 0$
53. $x^2y'' + 7xy' + 25y = 0$
54. $x^3y''' + 6x^2y'' + 4xy' = 0$
55. $x^3y''' - x^2y'' + xy' = 0$
56. $x^3y''' + 3x^2y'' + xy' = 0$
57. $x^3y''' - 3x^2y'' + xy' = 0$
58. $x^3y''' + 6x^2y'' + 7xy' + y = 0$

Application Approximate Solutions of Linear Equations

Go to `goo.gl/9JwK72` to download this application's computing resources including *Mathematica*/MATLAB.

To meet the needs of applications such as those of this section, polynomial-solving utilities are now a common feature of calculator and computer systems and can be used to solve a characteristic equation numerically even when no simple factorization is evident or even possible. For instance, suppose that we want to solve the homogeneous linear differential equation

$$y^{(3)} - 3y'' + y = 0 \tag{1}$$

with characteristic equation

$$r^3 - 3r^2 + 1 = 0. \tag{2}$$

A typical graphing calculator has a **`solve`** command that can be used to find the approximate roots of a polynomial equation. As indicated in Figs. 5.3.3 and 5.3.4, we find that the roots of Eq. (2) are given by $r \approx -0.5321$, 0.6527, and 2.8794. Some analogous computer algebra system commands are

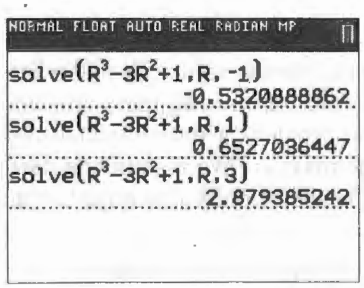

FIGURE 5.3.3. Solving the equation $r^3 - 3r^2 + 1 = 0$ with a TI-84 Plus CE calculator that requires an estimate of each root.

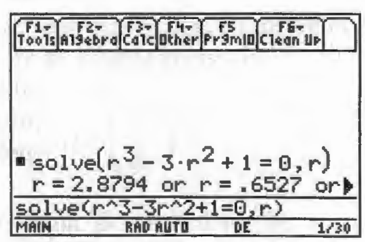

FIGURE 5.3.4. Solving the characteristic equation with a TI-89 calculator having a more sophisticated **`solve`** facility.

5.4 Problems

1. Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant 16 N/m.
2. Determine the period and frequency of the simple harmonic motion of a body of mass 0.75 kg on the end of a spring with spring constant 48 N/m.
3. A mass of 3 kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$ m/s. Find the amplitude, period, and frequency of the resulting motion.
4. A body with mass 250 g is attached to the end of a spring that is stretched 25 cm by a force of 9 N. At time $t = 0$ the body is pulled 1 m to the right, stretching the spring, and set in motion with an initial velocity of 5 m/s to the left. **(a)** Find $x(t)$ in the form $C\cos(\omega_0 t - \alpha)$. **(b)** Find the amplitude and period of motion of the body.

Simple Pendulum

In Problems 5 through 8, assume that the differential equation of a simple pendulum of length L is $L\theta'' + g\theta = 0$, where $g = GM/R^2$ is the gravitational acceleration at the location of the pendulum (at distance R from the center of the earth; M denotes the mass of the earth).

5. Two pendulums are of lengths L_1 and L_2 and—when located at the respective distances R_1 and R_2 from the center of the earth—have periods p_1 and p_2. Show that
$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}.$$
6. A certain pendulum keeps perfect time in Paris, where the radius of the earth is $R = 3956$ (mi). But this clock loses 2 min 40 s per day at a location on the equator. Use the result of Problem 5 to find the amount of the equatorial bulge of the earth.
7. A pendulum of length 100.10 in., located at a point at sea level where the radius of the earth is $R = 3960$ (mi), has the same period as does a pendulum of length 100.00 in. atop a nearby mountain. Use the result of Problem 5 to find the height of the mountain.
8. Most grandfather clocks have pendulums with adjustable lengths. One such clock loses 10 min per day when the length of its pendulum is 30 in. With what length pendulum will this clock keep perfect time?
9. Derive Eq. (5) describing the motion of a mass attached to the bottom of a vertically suspended spring. (*Suggestion:* First denote by $x(t)$ the displacement of the mass below the unstretched position of the spring; set up the differential equation for x. Then substitute $y = x - s_0$ in this differential equation.)
10. **Floating buoy** Consider a floating cylindrical buoy with radius r, height h, and uniform density $\rho \leqq 0.5$ (recall that the density of water is 1 g/cm^3). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t = 0$. Thereafter it is acted on by two forces: a downward gravitational force equal to its weight $mg = \rho\pi r^2 hg$ and (by Archimedes' principle of buoyancy) an upward force equal to the weight $\pi r^2 xg$ of water displaced, where $x = x(t)$ is the depth of the bottom of the buoy beneath the surface at time t (Fig. 5.4.12). Assume that friction is negligible. Conclude that the buoy undergoes simple harmonic motion around its equilibrium position $x_e = \rho h$ with period $p = 2\pi\sqrt{\rho h/g}$. Compute p and the amplitude of the motion if $\rho = 0.5$ g/cm^3, $h = 200$ cm, and $g = 980$ cm/s^2.
11. **Floating buoy** A cylindrical buoy weighing 100 lb (thus of mass $m = 3.125$ slugs in ft-lb-s (fps) units) floats in water with its axis vertical (as in Problem 10). When depressed slightly and released, it oscillates up and down four times every 10 s. Find the radius of the buoy.
12. **Hole through the earth** Assume that the earth is a solid sphere of uniform density, with mass M and radius $R = 3960$ (mi). For a particle of mass m *within* the earth at distance r from the center of the earth, the gravitational force attracting m toward the center is $F_r = -GM_r m/r^2$, where M_r is the mass of the part of the earth within a sphere of radius r (Fig. 5.4.13). **(a)** Show that $F_r = -GMmr/R^3$.

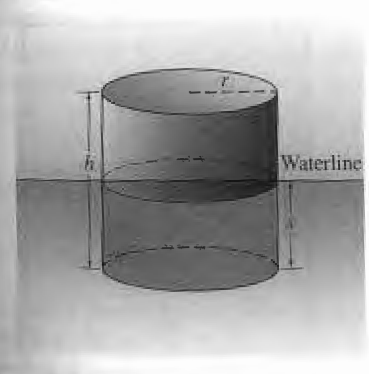

FIGURE 5.4.12. The buoy of Problem 10.

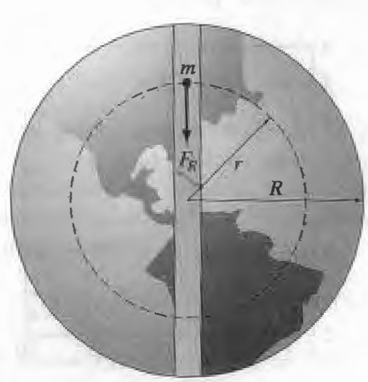

FIGURE 5.4.13. A mass m falling down a hole through the center of the earth (Problem 12).

(**b**) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass m be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time t, where we take $r < 0$ when the mass is "below" the center of the earth. Conclude from Newton's second law and part (a) that $r''(t) = -k^2 r(t)$, where $k^2 = GM/R^3 = g/R$. (**c**) Take $g = 32.2$ ft/s^2, and conclude from part (b) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 min. (**d**) Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (c). How do you explain the coincidence? Or *is* it a coincidence? (**e**) With what speed (in miles per hour) does the particle pass through the center of the earth? (**f**) Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (e). How do you explain the coincidence? Or *is* it a coincidence?

13. Suppose that the mass in a mass–spring–dashpot system with $m = 10$, $c = 9$, and $k = 2$ is set in motion with $x(0) = 0$ and $x'(0) = 5$. (**a**) Find the position function $x(t)$ and show that its graph looks as indicated in Fig. 5.4.14. (**b**) Find how far the mass moves to the right before starting back toward the origin.

14. Suppose that the mass in a mass–spring–dashpot system with $m = 25$, $c = 10$, and $k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$. (**a**) Find the position function $x(t)$ and show that its graph looks as indicated in Fig. 5.4.15. (**b**) Find the pseudoperiod of the oscillations and the equations of the "envelope curves" that are dashed in the figure.

Free Damped Motion

The remaining problems in this section deal with free damped motion. In Problems 15 through 21, a mass m is attached to both a spring (with given spring constant k) and a dashpot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Finally, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

15. $m = \frac{1}{2}, c = 3, k = 4; x_0 = 2, v_0 = 0$

16. $m = 3, c = 30, k = 63; x_0 = 2, v_0 = 2$

17. $m = 1, c = 8, k = 16; x_0 = 5, v_0 = -10$

18. $m = 2, c = 12, k = 50; x_0 = 0, v_0 = -8$

19. $m = 4, c = 20, k = 169; x_0 = 4, v_0 = 16$

20. $m = 2, c = 16, k = 40; x_0 = 5, v_0 = 4$

21. $m = 1, c = 10, k = 125; x_0 = 6, v_0 = 50$

22. Vertical damped motion A 12-lb weight (mass $m = 0.375$ slugs in fps units) is attached both to a vertically suspended spring that it stretches 6 in. and to a dashpot that provides 3 lb of resistance for every foot per second of velocity. (**a**) If the weight is pulled down 1 ft below its static equilibrium position and then released from rest at time $t = 0$, find its position function $x(t)$. (**b**) Find the frequency, time-varying amplitude, and phase angle of the motion.

23. Car suspension This problem deals with a highly simplified model of a car of weight 3200 lb (mass $m = 100$ slugs in fps units). Assume that the suspension system acts like a single spring and its shock absorbers like a single dashpot, so that its vertical vibrations satisfy Eq. (4) with appropriate values of the coefficients. (**a**) Find the stiffness coefficient k of the spring if the car undergoes free vibrations at 80 cycles per minute (cycles/min) when its shock absorbers are disconnected. (**b**) With the shock absorbers connected, the car is set into vibration by driving it over a bump, and the resulting damped vibrations have a frequency of 78 cycles/min. After how long will the time-varying amplitude be 1% of its initial value?

Problems 24 through 34 deal with a mass–spring–dashpot system having position function $x(t)$ satisfying Eq. (4). We write

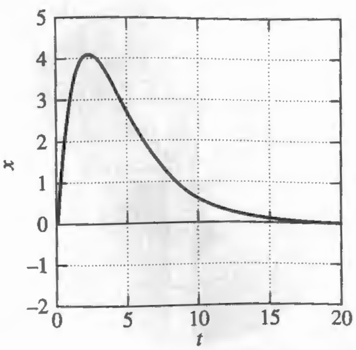

FIGURE 5.4.14. The position function $x(t)$ of Problem 13.

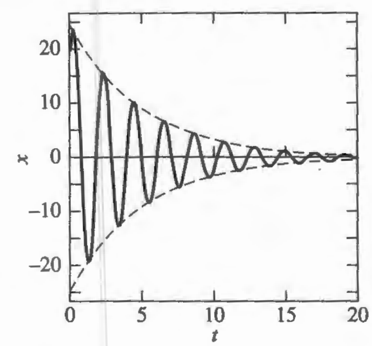

FIGURE 5.4.15. The position function $x(t)$ of Problem 14.

$x_0 = x(0)$ and $v_0 = x'(0)$ and recall that $p = c/(2m)$, $\omega_0^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. The system is critically damped, overdamped, or underdamped, as specified in each problem.

24. (Critically damped) Show in this case that

$$x(t) = (x_0 + v_0 t + p x_0 t)e^{-pt}.$$

25. (Critically damped) Deduce from Problem 24 that the mass passes through $x = 0$ at some instant $t > 0$ if and only if x_0 and $v_0 + p x_0$ have opposite signs.

26. (Critically damped) Deduce from Problem 24 that $x(t)$ has a local maximum or minimum at some instant $t > 0$ if and only if v_0 and $v_0 + p x_0$ have the same sign.

27. (Overdamped) Show in this case that

$$x(t) = \frac{1}{2\gamma}\left[(v_0 - r_2 x_0)e^{r_1 t} - (v_0 - r_1 x_0)e^{r_2 t}\right],$$

where $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$ and $\gamma = (r_1 - r_2)/2 > 0$.

28. (Overdamped) If $x_0 = 0$, deduce from Problem 27 that

$$x(t) = \frac{v_0}{\gamma}e^{-pt}\sinh \gamma t.$$

29. (Overdamped) Prove that in this case the mass can pass through its equilibrium position $x = 0$ at most once.

30. (Underdamped) Show that in this case

$$x(t) = e^{-pt}\left(x_0 \cos \omega_1 t + \frac{v_0 + p x_0}{\omega_1}\sin \omega_1 t\right).$$

31. (Underdamped) If the damping constant c is small in comparison with $\sqrt{8mk}$, apply the binomial series to show that

$$\omega_1 \approx \omega_0\left(1 - \frac{c^2}{8mk}\right).$$

32. (Underdamped) Show that the local maxima and minima of

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$$

occur where

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}.$$

Conclude that $t_2 - t_1 = 2\pi/\omega_1$ if two consecutive maxima occur at times t_1 and t_2.

33. (Underdamped) Let x_1 and x_2 be two consecutive local maximum values of $x(t)$. Deduce from the result of Problem 32 that

$$\ln \frac{x_1}{x_2} = \frac{2\pi p}{\omega_1}.$$

The constant $\Delta = 2\pi p/\omega_1$ is called the **logarithmic decrement** of the oscillation. Note also that $c = m\omega_1 \Delta/\pi$ because $p = c/(2m)$.

Note: The result of Problem 33 provides an accurate method for measuring the viscosity of a fluid, which is an important parameter in fluid dynamics but is not easy to measure directly. According to Stokes's drag law, a spherical body of radius a moving at a (relatively slow) speed through a fluid of viscosity μ experiences a resistive force $F_R = 6\pi\mu a v$. Thus if a spherical mass on a spring is immersed in the fluid and set in motion, this drag resistance damps its oscillations with damping constant $c = 6\pi a\mu$. The frequency ω_1 and logarithmic decrement Δ of the oscillations can be measured by direct observation. The final formula in Problem 33 then gives c and hence the viscosity of the fluid.

34. (Underdamped) A body weighing 100 lb (mass $m = 3.125$ slugs in fps units) is oscillating attached to a spring and a dashpot. Its first two maximum displacements of 6.73 in. and 1.46 in. are observed to occur at times 0.34 s and 1.17 s, respectively. Compute the damping constant (in pound-seconds per foot) and spring constant (in pounds per foot).

Differential Equations and Determinism

Given a mass m, a dashpot constant c, and a spring constant k, Theorem 2 of Section 5.1 implies that the equation

$$mx'' + cx' + kx = 0 \tag{26}$$

has a unique solution for $t \geqq 0$ satisfying given initial conditions $x(0) = x_0$, $x'(0) = v_0$. Thus the future motion of an ideal mass–spring–dashpot system is completely determined by the differential equation and the initial conditions. Of course in a real physical system it is impossible to measure the parameters m, c, and k precisely. Problems 35 through 38 explore the resulting uncertainty in predicting the future behavior of a physical system.

35. Suppose that $m = 1$, $c = 2$, and $k = 1$ in Eq. (26). Show that the solution with $x(0) = 0$ and $x'(0) = 1$ is

$$x_1(t) = te^{-t}.$$

36. Suppose that $m = 1$ and $c = 2$ but $k = 1 - 10^{-2n}$. Show that the solution of Eq. (26) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_2(t) = 10^n e^{-t}\sinh 10^{-n} t.$$

37. Suppose that $m = 1$ and $c = 2$ but that $k = 1 + 10^{-2n}$. Show that the solution of Eq. (26) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_3(t) = 10^n e^{-t}\sin 10^{-n} t.$$

38. Whereas the graphs of $x_1(t)$ and $x_2(t)$ resemble those shown in Figs. 5.4.7 and 5.4.8, the graph of $x_3(t)$ exhibits damped oscillations like those illustrated in Fig. 5.4.9, but with a very long pseudoperiod. Nevertheless, show that for each fixed $t > 0$ it is true that

$$\lim_{n\to\infty} x_2(t) = \lim_{n\to\infty} x_3(t) = x_1(t).$$

Conclude that *on a given finite time interval* the three solutions are in "practical" agreement if n is sufficiently large.

5.5 Problems

In Problems 1 through 20, find a particular solution y_p of the given equation. In all these problems, primes denote derivatives with respect to x.

1. $y'' + 16y = e^{3x}$
2. $y'' - y' - 2y = 3x + 4$
3. $y'' - y' - 6y = 2\sin 3x$
4. $4y'' + 4y' + y = 3xe^x$
5. $y'' + y' + y = \sin^2 x$
6. $2y'' + 4y' + 7y = x^2$
7. $y'' - 4y = \sinh x$
8. $y'' - 4y = \cosh 2x$
9. $y'' + 2y' - 3y = 1 + xe^x$
10. $y'' + 9y = 2\cos 3x + 3\sin 3x$
11. $y^{(3)} + 4y' = 3x - 1$
12. $y^{(3)} + y' = 2 - \sin x$
13. $y'' + 2y' + 5y = e^x \sin x$
14. $y^{(4)} - 2y'' + y = xe^x$
15. $y^{(5)} + 5y^{(4)} - y = 17$
16. $y'' + 9y = 2x^2e^{3x} + 5$
17. $y'' + y = \sin x + x\cos x$
18. $y^{(4)} - 5y'' + 4y = e^x - xe^{2x}$
19. $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$
20. $y^{(3)} - y = e^x + 7$

In Problems 21 through 30, set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

21. $y'' - 2y' + 2y = e^x \sin x$
22. $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$
23. $y'' + 4y = 3x\cos 2x$
24. $y^{(3)} - y'' - 12y' = x - 2xe^{-3x}$
25. $y'' + 3y' + 2y = x(e^{-x} - e^{-2x})$
26. $y'' - 6y' + 13y = xe^{3x}\sin 2x$
27. $y^{(4)} + 5y'' + 4y = \sin x + \cos 2x$
28. $y^{(4)} + 9y'' = (x^2 + 1)\sin 3x$
29. $(D - 1)^3(D^2 - 4)y = xe^x + e^{2x} + e^{-2x}$
30. $y^{(4)} - 2y'' + y = x^2\cos x$

Solve the initial value problems in Problems 31 through 40.

31. $y'' + 4y = 2x;\ y(0) = 1,\ y'(0) = 2$
32. $y'' + 3y' + 2y = e^x;\ y(0) = 0,\ y'(0) = 3$
33. $y'' + 9y = \sin 2x;\ y(0) = 1,\ y'(0) = 0$
34. $y'' + y = \cos x;\ y(0) = 1,\ y'(0) = -1$
35. $y'' - 2y' + 2y = x + 1;\ y(0) = 3,\ y'(0) = 0$
36. $y^{(4)} - 4y'' = x^2;\ y(0) = y'(0) = 1,\ y''(0) = y^{(3)}(0) = -1$
37. $y^{(3)} - 2y'' + y' = 1 + xe^x;\ y(0) = y'(0) = 0,\ y''(0) = 1$
38. $y'' - 2y' + 2y = \sin 3x;\ y(0) = 2,\ y'(0) = 0$
39. $y^{(3)} + y'' = x + e^{-x};\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1$
40. $y^{(4)} - y = 5;\ y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

41. Find a particular solution of the equation

$$y^{(4)} - y^{(3)} - y'' - y' - 2y = 8x^5.$$

42. Find the solution of the initial value problem consisting of the differential equation of Problem 41 and the initial conditions

$$y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

43. **(a)** Write

$$\cos 3x + i\sin 3x = e^{3ix} = (\cos x + i\sin x)^3$$

by Euler's formula, expand, and equate real and imaginary parts to derive the identities

$$\cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x,$$
$$\sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x.$$

(b) Use the result of part (a) to find a general solution of

$$y'' + 4y = \cos^3 x.$$

Use trigonometric identities to find general solutions of the equations in Problems 44 through 46.

44. $y'' + y' + y = \sin x \sin 3x$
45. $y'' + 9y = \sin^4 x$
46. $y'' + y = x\cos^3 x$

In Problems 47 through 56, use the method of variation of parameters to find a particular solution of the given differential equation.

47. $y'' + 3y' + 2y = 4e^x$
48. $y'' - 2y' - 8y = 3e^{-2x}$
49. $y'' - 4y' + 4y = 2e^{2x}$
50. $y'' - 4y = \sinh 2x$
51. $y'' + 4y = \cos 3x$
52. $y'' + 9y = \sin 3x$
53. $y'' + 9y = 2\sec 3x$
54. $y'' + y = \csc^2 x$
55. $y'' + 4y = \sin^2 x$
56. $y'' - 4y = xe^x$

57. You can verify by substitution that $y_c = c_1x + c_2x^{-1}$ is a complementary function for the nonhomogeneous second-order equation

$$x^2y'' + xy' - y = 72x^5.$$

But before applying the method of variation of parameters, you must first divide this equation by its leading coefficient x^2 to rewrite it in the standard form

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 72x^3.$$

Thus $f(x) = 72x^3$ in Eq. (22). Now proceed to solve the equations in (31) and thereby derive the particular solution $y_p = 3x^5$.

In Problems 58 through 62, a nonhomogeneous second-order linear equation and a complementary function y_c are given. Apply the method of Problem 57 to find a particular solution of the equation.

58. $x^2y'' - 4xy' + 6y = x^3;\ y_c = c_1x^2 + c_2x^3$
59. $x^2y'' - 3xy' + 4y = x^4;\ y_c = x^2(c_1 + c_2\ln x)$
60. $4x^2y'' - 4xy' + 3y = 8x^{4/3};\ y_c = c_1x + c_2x^{3/4}$
61. $x^2y'' + xy' + y = \ln x;\ y_c = c_1\cos(\ln x) + c_2\sin(\ln x)$
62. $(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1;\ y_c = c_1x + c_2(1 + x^2)$
63. Carry out the solution process indicated in the text to derive the variation of parameters formula in (33) from Eqs. (31) and (32).
64. Apply the variation of parameters formula in (33) to find the particular solution $y_p(x) = -x\cos x$ of the nonhomogeneous equation $y'' + y = 2\sin x$.

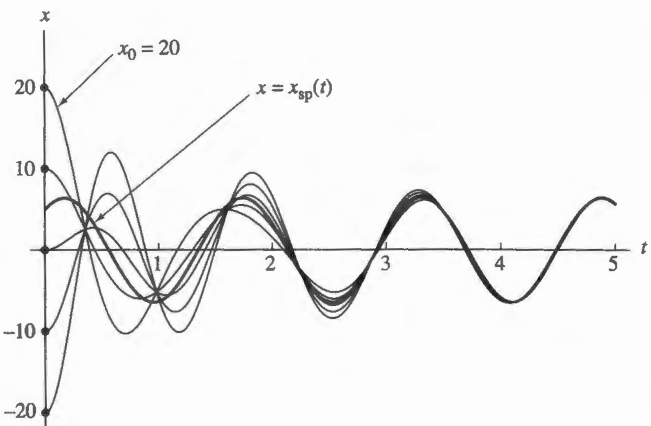

FIGURE 5.6.8. Solutions of the initial value problem in (24) with $x_0 = -20, -10, 0, 10$, and 20.

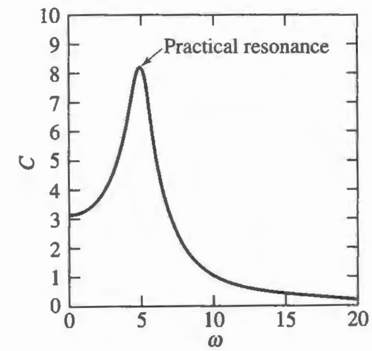

FIGURE 5.6.9. Plot of amplitude C versus external frequency ω.

Problems

In Problems 1 through 6, express the solution of the given initial value problem as a sum of two oscillations as in Eq. (8). Throughout, primes denote derivatives with respect to time t. In Problems 1–4, graph the solution function $x(t)$ in such a way that you can identify and label (as in Fig. 5.6.2) its period.

1. $x'' + 9x = 10\cos 2t$; $x(0) = x'(0) = 0$
2. $x'' + 4x = 5\sin 3t$; $x(0) = x'(0) = 0$
3. $x'' + 100x = 225\cos 5t + 300\sin 5t$; $x(0) = 375$, $x'(0) = 0$
4. $x'' + 25x = 90\cos 4t$; $x(0) = 0$, $x'(0) = 90$
5. $mx'' + kx = F_0\cos\omega t$ with $\omega \neq \omega_0$; $x(0) = x_0$, $x'(0) = 0$
6. $mx'' + kx = F_0\cos\omega t$ with $\omega = \omega_0$; $x(0) = 0$, $x'(0) = v_0$

In each of Problems 7 through 10, find the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ of the given equation $mx'' + cx' + kx = F(t)$ with periodic forcing function $F(t)$ of frequency ω. Then graph $x_{sp}(t)$ together with (for comparison) the adjusted forcing function $F_1(t) = F(t)/m\omega$.

7. $x'' + 4x' + 4x = 10\cos 3t$
8. $x'' + 3x' + 5x = -4\cos 5t$
9. $2x'' + 2x' + x = 3\sin 10t$
10. $x'' + 3x' + 3x = 8\cos 10t + 6\sin 10t$

In each of Problems 11 through 14, find and plot both the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ of the given differential equation and the actual solution $x(t) = x_{sp}(t) + x_{tr}(t)$ that satisfies the given initial conditions.

11. $x'' + 4x' + 5x = 10\cos 3t$; $x(0) = x'(0) = 0$
12. $x'' + 6x' + 13x = 10\sin 5t$; $x(0) = x'(0) = 0$
13. $x'' + 2x' + 26x = 600\cos 10t$; $x(0) = 10$, $x'(0) = 0$
14. $x'' + 8x' + 25x = 200\cos t + 520\sin t$; $x(0) = -30$, $x'(0) = -10$

Each of Problems 15 through 18 gives the parameters for a forced [redacted]-spring–dashpot system with equation $mx'' + cx' + kx = F_0\cos\omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency ω. Sketch the graph of $C(\omega)$ and find the practical resonance frequency ω (if any).

15. $m = 1$, $c = 2$, $k = 2$, $F_0 = 2$
16. $m = 1$, $c = 4$, $k = 5$, $F_0 = 10$
17. $m = 1$, $c = 6$, $k = 45$, $F_0 = 50$
18. $m = 1$, $c = 10$, $k = 650$, $F_0 = 100$
19. A mass weighing 100 lb (mass $m = 3.125$ slugs in fps units) is attached to the end of a spring that is stretched 1 in. by a force of 100 lb. A force $F_0\cos\omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillations occur? Neglect damping.
20. A front-loading washing machine is mounted on a thick rubber pad that acts like a spring; the weight $W = mg$ (with $g = 9.8$ m/s^2) of the machine depresses the pad exactly 0.5 cm. When its rotor spins at ω radians per second, the rotor exerts a vertical force $F_0\cos\omega t$ newtons on the machine. At what speed (in revolutions per minute) will resonance vibrations occur? Neglect friction.
21. **Pendulum-spring system** Figure 5.6.10 shows a mass m on the end of a pendulum (of length L) also attached to a horizontal spring (with constant k). Assume small oscillations of m so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency ω_0 of motion of the mass in terms of L, k, m, and the gravitational constant g.
22. **Pulley-spring system** A mass m hangs on the end of a cord around a pulley of radius a and moment of inertia I, as shown in Fig. 5.6.11. The rim of the pulley is attached to a spring (with constant k). Assume small oscillations so that the spring remains essentially horizontal and neglect friction. Find the natural circular frequency of the system in terms of m, a, k, I, and g.

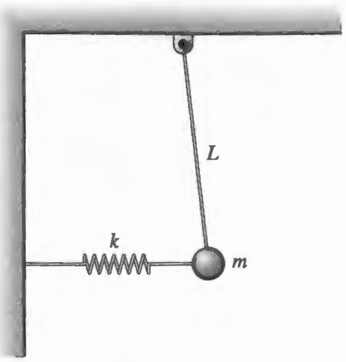

FIGURE 5.6.10. The pendulum-and-spring system of Problem 21.

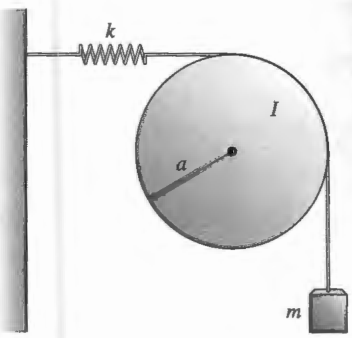

FIGURE 5.6.11. The mass–spring–pulley system of Problem 22.

23. Earthquake A building consists of two floors. The first floor is attached rigidly to the ground, and the second floor is of mass $m = 1000$ slugs (fps units) and weighs 16 tons (32,000 lb). The elastic frame of the building behaves as a spring that resists horizontal displacements of the second floor; it requires a horizontal force of 5 tons to displace the second floor a distance of 1 ft. Assume that in an earthquake the ground oscillates horizontally with amplitude A_0 and circular frequency ω, resulting in an external horizontal force $F(t) = mA_0\omega^2 \sin \omega t$ on the second floor. **(a)** What is the natural frequency (in hertz) of oscillations of the second floor? **(b)** If the ground undergoes one oscillation every 2.25 s with an amplitude of 3 in., what is the amplitude of the resulting forced oscillations of the second floor?

24. A mass on a spring without damping is acted on by the external force $F(t) = F_0 \cos^3 \omega t$. Show that there are *two* values of ω for which resonance occurs, and find both.

25. Derive the steady periodic solution of

$$mx'' + cx' + kx = F_0 \sin \omega t.$$

In particular, show that it is what one would expect—the same as the formula in (20) with the same values of C and ω, except with $\sin(\omega t - \alpha)$ in place of $\cos(\omega t - \alpha)$.

26. Given the differential equation

$$mx'' + cx' + kx = E_0 \cos \omega t + F_0 \sin \omega t$$

—with both cosine and sine forcing terms—derive the steady periodic solution

$$x_{sp}(t) = \frac{\sqrt{E_0^2 + F_0^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \alpha - \beta),$$

where α is defined in Eq. (22) and $\beta = \tan^{-1}(F_0/E_0)$. (*Suggestion:* Add the steady periodic solutions separately corresponding to $E_0 \cos \omega t$ and $F_0 \sin \omega t$ (see Problem 25).)

27. According to Eq. (21), the amplitude of forced steady periodic oscillations for the system $mx'' + cx' + kx = F_0 \cos \omega t$ is given by

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(a) If $c \geqq c_{cr}/\sqrt{2}$, where $c_{cr} = \sqrt{4km}$, show that C steadily decreases as ω increases. **(b)** If $c < c_{cr}/\sqrt{2}$, show that C attains a maximum value (practical resonance) when

$$\omega = \omega_m = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} < \omega_0 = \sqrt{\frac{k}{m}}.$$

28. As indicated by the cart-with-flywheel example discussed in this section, an unbalanced rotating machine part typically results in a force having amplitude proportional to the *square* of the frequency ω. **(a)** Show that the amplitude of the steady periodic solution of the differential equation

$$mx'' + cx' + kx = mA\omega^2 \cos \omega t$$

(with a forcing term similar to that in Eq. (17)) is given by

$$C(\omega) = \frac{mA\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(b) Suppose that $c^2 < 2mk$. Show that the maximum amplitude occurs at the frequency ω_m given by

$$\omega_m = \sqrt{\frac{k}{m}\left(\frac{2mk}{2mk - c^2}\right)}.$$

Thus the resonance frequency in this case is *larger* (in contrast with the result of Problem 27) than the natural frequency $\omega_0 = \sqrt{k/m}$. (*Suggestion:* Maximize the *square* of C.)

Automobile Vibrations

Problems 29 and 30 deal further with the car of Example 5. Its upward displacement function satisfies the equation

$mx'' + cx' + kx = cy' + ky$ when the shock absorber is connected (so that $c > 0$). With $y = a \sin \omega t$ for the road surface, this differential equation becomes

$$mx'' + cx' + kx = E_0 \cos \omega t + F_0 \sin \omega t$$

where $E_0 = c\omega a$ and $F_0 = ka$.

29. Apply the result of Problem 26 to show that the amplitude C of the resulting steady periodic oscillation for the car is given by

$$C = \frac{a\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

Because $\omega = 2\pi v/L$ when the car is moving with velocity v, this gives C as a function of v.

30. Figure 5.6.12 shows the graph of the amplitude function $C(\omega)$ using the numerical data given in Example 5 (including $c = 3000$ N·s/m). It indicates that, as the car accelerates gradually from rest, it initially oscillates with amplitude slightly over 5 cm. Maximum resonance vibrations with amplitude about 14 cm occur around 32 mi/h, but then subside to more tolerable levels at high speeds. Verify these graphically based conclusions by analyzing the function $C(\omega)$. In particular, find the practical resonance frequency and the corresponding amplitude.

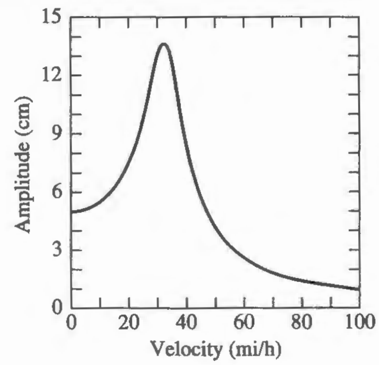

FIGURE 5.6.12. Amplitude of vibrations of the car on a washboard surface.

Application Forced Vibrations

Go to goo.gl/12hK16 to download this application's computing resources including Mathematica/MATLAB.

Here we investigate forced vibrations of the mass–spring–dashpot system with equation

$$mx'' + cx' + kx = F(t). \tag{1}$$

To simplify the notation, let's take $m = p^2$, $c = 2p$, and $k = p^2q^2 + 1$, where $p > 0$ and $q > 0$. Then the complementary function of Eq. (1) is

$$x_c(t) = e^{-t/p}(c_1 \cos qt + c_2 \sin qt). \tag{2}$$

We will take $p = 5$, $q = 3$, and thus investigate the transient and steady periodic solutions corresponding to

$$25x'' + 10x' + 226x = F(t), \quad x(0) = 0, \quad x'(0) = 0 \tag{3}$$

with several illustrative possibilities for the external force $F(t)$. For your personal investigations to carry out similarly, you might select integers p and q with $6 \leqq p \leqq 9$ and $2 \leqq q \leqq 5$.

INVESTIGATION 1: With periodic external force $F(t) = 901 \cos 3t$, the MATLAB commands

```
x = dsolve('25*D2x+10*Dx+226*x=901*cos(3*t)',
      'x(0)=0, Dx(0)=0');
x = simple(x);
syms t, xsp = cos(3*t) + 30*sin(3*t);
ezplot(x, [0  6*pi]),hold on
ezplot(xsp, [0  6*pi])
```

produce the plot shown in Fig. 5.6.13. We see the (transient plus steady periodic) solution

$$x(t) = \cos 3t + 30 \sin 3t + e^{-t/5}\left(-\cos 3t - \tfrac{451}{15} \sin 3t\right)$$

rapidly "building up" to the steady periodic oscillation $x_{sp}(t) = \cos 3t + 30 \sin 3t$.

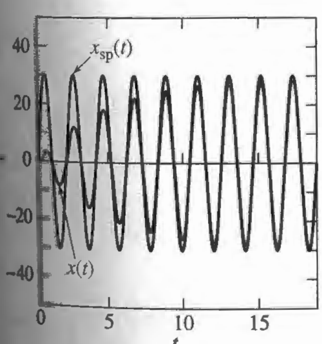

FIGURE 5.6.13. The solution $x(t) = x_{tr}(t) + x_{sp}(t)$ and the steady periodic solution $x(t) = x_{sp}(t)$ with periodic external force $F(t) = 901 \cos 3t$.