CASE 1: $\lambda = 2$. The system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which reduces to the single equation $2x - 2y + z = 0$. This equation obviously has a dimensional solution space. With $y = 1$ and $z = 0$, we get $x = 1$ and, hence, obtain the ba eigenvector $\mathbf{v}_1 = [\,1 \quad 1 \quad 0\,]^T$. With $y = 0$ and $z = 2$, we get $x = -1$ and, hence, the ba eigenvector $\mathbf{v}_2 = [\,-1 \quad 0 \quad 2\,]^T$. The 2-dimensional eigenspace of $\mathbf{A}$ associated with t repeated eigenvalue $\lambda = 2$ has basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

CASE 2: $\lambda = 3$. The system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last equation here implies that $x = y$, and then each of the first two equations yield $x = y = z$. It follows that the eigenspace of $\mathbf{A}$ associated with $\lambda = 3$ is 1-dimensional and has $\mathbf{v}_3 = [\,1 \quad 1 \quad 1\,]^T$ as a basis eigenvector. ■

Remark The typical higher-degree polynomial is not so easy to factor as the one in Example 7. Hence a numerical technique such as Newton's method is often needed to solve the characteristic equation. Moreover, for an $n \times n$ matrix with n greater than about 4, the amount of labor required to find the characteristic equation by expanding the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is generally prohibitive; because of the presence of the variable λ, row and column elimination methods do not work as they do with numerical determinants. Consequently, specialized techniques, beyond the scope of the present discussion, are often required to find the eigenvalues and eigenvectors of the large matrices that occur in many applications. Problems 40 and 41 at the end of this section outline a numerical technique that sometimes is useful with matrices of moderate size. ■

6.1 Problems

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix **A**. *Find a basis for each eigenspace of dimension 2 or larger.*

1. $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ **3.** $\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ **5.** $\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix}$ **6.** $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

7. $\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$ **8.** $\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix}$ **9.** $\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix}$ **12.** $\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix}$ **14.** $\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix}$

15. $\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix}$

21. $\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ **22.** $\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **24.** $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ **26.** $\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix}$

Find the complex conjugate eigenvalues and corresponding eigenvectors of the matrices given in Problems 27 through 32.

27. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **28.** $\mathbf{A} = \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$

29. $\mathbf{A} = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$ **30.** $\mathbf{A} = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}$

31. $\mathbf{A} = \Big[\begin{matrix} 0 \\ -6 \end{matrix}$

33. Suppose th ciated eige that λ^n is vector **v**.

34. Show that and only i ated eigen

35. **(a)** Suppo istic e eigen
(b) Give $\mathbf{A}^T$ d

36. Show that its diagon

37. Suppose written as constant appropria

Problems 38 trix and expl characteristic

38. If $\mathbf{A} = [$ is define

the sum proved t $(-1)^{n-1}$ of a $2 \times$

39. Suppose ues $\lambda_1,$ Problem

λ_1

40. Accordi characte

6.2 I

31. $\mathbf{A} = \begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix}$ **32.** $\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix}$

33. Suppose that λ is an eigenvalue of the matrix $\mathbf{A}$ with associated eigenvector $\mathbf{v}$ and that n is a positive integer. Show that λ^n is an eigenvalue of $\mathbf{A}^n$ with associated eigenvector $\mathbf{v}$.

34. Show that λ is an eigenvalue of the invertible matrix $\mathbf{A}$ if and only if λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. Are the associated eigenvectors the same?

35. **(a)** Suppose that $\mathbf{A}$ is a square matrix. Use the characteristic equation to show that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.
(b) Give an example of a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}$ and $\mathbf{A}^T$ do not have the same eigenvectors.

36. Show that the eigenvalues of a triangular $n \times n$ matrix are its diagonal elements.

37. Suppose that the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is written as a polynomial equation [Eq. (5)]. Show that the constant term is $c_0 = \det \mathbf{A}$. *Suggestion*: Substitute an appropriate value for λ.

Problems 38 through 42 introduce the trace of a square matrix and explore its connections with the determinant and the characteristic polynomial of the matrix.

38. If $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $n \times n$ matrix, then the **trace** $\operatorname{Tr}\mathbf{A}$ of $\mathbf{A}$ is defined to be

$$\operatorname{Tr}\mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn},$$

the sum of the diagonal elements of $\mathbf{A}$. It can be proved that the coefficient of λ^{n-1} in Eq. (5) is $c_{n-1} = (-1)^{n-1}(\operatorname{Tr}\mathbf{A})$. Show explicitly that this is true in the case of a 2×2 matrix.

39. Suppose that the $n \times n$ matrix $\mathbf{A}$ has n (real) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Assuming the general result stated in Problem 38, prove that

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &= \operatorname{Tr}\mathbf{A} \\ &= a_{11} + a_{22} + \cdots + a_{nn}. \end{aligned}$$

40. According to the results stated in Problems 37 and 38, the characteristic polynomial

$$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$$

of a 3×3 matrix $\mathbf{A}$ is given by

$$p(\lambda) = -\lambda^3 + (\operatorname{Tr}\mathbf{A})\lambda^2 + c_1\lambda + (\det \mathbf{A}).$$

The remaining coefficient c_1 can be found by substituting $\lambda = 1$ and then calculating the two determinants $|\mathbf{A}|$ and $p(1) = |\mathbf{A} - \mathbf{I}|$. Use this method to find the characteristic equation, eigenvalues, and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix}.$$

41. According to the results stated in Problems 37 and 38, the characteristic polynomial

$$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$$

of a 4×4 matrix $\mathbf{A}$ is given by

$$p(\lambda) = \lambda^4 - (\operatorname{Tr}\mathbf{A})\lambda^3 + c_2\lambda^2 + c_1\lambda + (\det \mathbf{A}).$$

The remaining coefficients c_1 and c_2 can be found by substituting $\lambda = \pm 1$ and calculating the three determinants $|\mathbf{A}|$, $p(1) = |\mathbf{A} - \mathbf{I}|$, and $p(-1) = |\mathbf{A} + \mathbf{I}|$. Use this method to find the characteristic equation, eigenvalues, and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix}.$$

42. Combine ideas from Problems 37–40 to show explicitly that
(a) If $\mathbf{A}$ is a 2×2 matrix then $\det(\mathbf{A} - \lambda\mathbf{I}) = (-\lambda)^2 + \operatorname{Tr}(\mathbf{A})(-\lambda) + \det(\mathbf{A})$.
(b) If $\mathbf{A}$ is a 3×3 matrix then $\det(\mathbf{A} - \lambda\mathbf{I}) = (-\lambda)^3 + \operatorname{Tr}(\mathbf{A})(-\lambda)^2 + c_1(-\lambda) + \det(\mathbf{A})$, where c_1 is the sum of the minors of the diagonal elements of $\mathbf{A}$. (Recall from Section 3.6 the distinction between minors and cofactors.)

6.2 Diagonalization of Matrices

Given an $n \times n$ matrix $\mathbf{A}$, we may ask *how many* linearly independent eigenvectors the matrix $\mathbf{A}$ has. In Section 6.1, we saw several examples (with $n = 2$ and $n = 3$) in which the $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors—the largest possible number. By contrast, in Example 5 of Section 6.1, we saw that the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

has the single eigenvalue $\lambda = 2$ corresponding to the single eigenvector $\mathbf{v} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

Something very nice happens when the $n \times n$ matrix $\mathbf{A}$ does have n linearly independent eigenvectors. Suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessar-