The first component of this column vector is

$$y_p = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \int \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix} dt = -y_1 \int \frac{y_2 f}{W}\, dt + y_2 \int \frac{y_1 f}{W}\, dt.$$

If, finally, we supply the independent variable t throughout, the final result on the right-hand side here is simply the variation of parameters formula in Eq. (33) of Section 5.5 (where, however, the independent variable is denoted by x).

8.2 Problems

Apply the method of undetermined coefficients to find a particular solution of each of the systems in Problems 1 through 14. If initial conditions are given, find the particular solution that satisfies these conditions. Primes denote derivatives with respect to t.

1. $x' = x + 2y + 3,\ y' = 2x + y - 2$
2. $x' = 2x + 3y + 5,\ y' = 2x + y - 2t$
3. $x' = 3x + 4y,\ y' = 3x + 2y + t^2;\ x(0) = y(0) = 0$
4. $x' = 4x + y + e^t,\ y' = 6x - y - e^t;\ x(0) = y(0) = 1$
5. $x' = 6x - 7y + 10,\ y' = x - 2y - 2e^{-t}$
6. $x' = 9x + y + 2e^t,\ y' = -8x - 2y + te^t$
7. $x' = -3x + 4y + \sin t,\ y' = 6x - 5y;\ x(0) = 1,\ y(0) = 0$
8. $x' = x - 5y + 2\sin t,\ y' = x - y - 3\cos t$
9. $x' = x - 5y + \cos 2t,\ y' = x - y$
10. $x' = x - 2y,\ y' = 2x - y + e^t \sin t$
11. $x' = 2x + 4y + 2,\ y' = x + 2y + 3;\ x(0) = 1,\ y(0) = -1$
12. $x' = x + y + 2t,\ y' = x + y - 2t$
13. $x' = 2x + y + 2e^t,\ y' = x + 2y - 3e^t$
14. $x' = 2x + y + 1,\ y' = 4x + 2y + e^{4t}$

Two Brine Tanks

Problems 15 and 16 are similar to Example 2, but with two brine tanks (having volumes V_1 and V_2 gallons as in Fig. 8.2.1) instead of three tanks. Each tank initially contains fresh water, and the inflow to tank 1 at the rate of r gallons per minute has a salt concentration of c_0 pounds per gallon. (a) Find the amounts $x_1(t)$ and $x_2(t)$ of salt in the two tanks after t minutes. (b) Find the limiting (long-term) amount of salt in each tank. (c) Find how long it takes for each tank to reach a salt concentration of 1 lb/gal.

15. $V_1 = 100,\ V_2 = 200,\ r = 10,\ c_0 = 2$
16. $V_1 = 200,\ V_2 = 100,\ r = 10,\ c_0 = 3$

In Problems 17 through 34, use the method of variation of parameters (and perhaps a computer algebra system) to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(a) = \mathbf{x}_a.$$

In each problem we provide the matrix exponential $e^{\mathbf{A}t}$ as provided by a computer algebra system.

17. $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 60 \\ 90 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$
$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}$
18. Repeat Problem 17, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 100t \\ -50t \end{bmatrix}$.
19. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 180t \\ 90 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$
$e^{\mathbf{A}t} = \frac{1}{5}\begin{bmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{bmatrix}$
20. Repeat Problem 19, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 75e^{2t} \\ 0 \end{bmatrix}$.
21. $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 18e^{2t} \\ 30e^{2t} \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$
$e^{\mathbf{A}t} = \frac{1}{4}\begin{bmatrix} -e^{-t} + 5e^{3t} & e^{-t} - e^{3t} \\ -5e^{-t} + 5e^{3t} & 5e^{-t} - e^{3t} \end{bmatrix}$
22. Repeat Problem 21, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 28e^{-t} \\ 20e^{3t} \end{bmatrix}$.
23. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 7 \\ 5 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$
$e^{\mathbf{A}t} = \begin{bmatrix} 1 + 3t & -t \\ 9t & 1 - 3t \end{bmatrix}$
24. Repeat Problem 23, but with $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$ and $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.
25. $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 4t \\ 1 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$
$e^{\mathbf{A}t} = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}$
26. Repeat Problem 25, but with $\mathbf{f}(t) = \begin{bmatrix} 4\cos t \\ 6\sin t \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.
27. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 36t^2 \\ 6t \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$
$e^{\mathbf{A}t} = \begin{bmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{bmatrix}$

28. Repeat Problem 27, but with $\mathbf{f}(t) = \begin{bmatrix} 4\ln t \\ t^{-1} \end{bmatrix}$ and $\mathbf{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

29. $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

30. $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t\cos 2t \\ t\sin 2t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$$

31. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 6e^t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 2te^t & (3t+2t^2)e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}$$

32. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 2e^{2t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 3te^t & (-13-9t)e^t + 13e^{2t} \\ 0 & e^t & -3e^t + 3e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

33. $\mathbf{A} = \begin{bmatrix} 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{f}(t) = 30\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 4t & 8t+6t^2 & 32t^2+8t^3 \\ 0 & 1 & 3t & 8t+6t^2 \\ 0 & 0 & 1 & 4t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

34. $\mathbf{A} = \begin{bmatrix} 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 6t \\ 0 \\ e^{2t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 4t & 4(-1+e^{2t}) & 16t(-1+e^{2t}) \\ 0 & 1 & 0 & 4(-1+e^{2t}) \\ 0 & 0 & e^{2t} & 4te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

8.2 Application Automated Variation of Parameters

Go to goo.gl/EEXGSc to download this application's computing resources including *Maple*/*Mathematica*/MATLAB.

The application of the variation of parameters formula in Eq. (28) encourages so mechanical an approach as to encourage especially the use of a computer algebra system. The following *Mathematica* commands were used to check the results in Example 4 of this section.

```
A = {{4,2}, {3,-1}};
x0 = {{7}, {3}};
f[t_] := {{-15 t Exp[-2t]},{-4 t Exp[-2t]}};
exp[A_] := MatrixExp[A]
x = exp[A*t].(x0 + Integrate[exp[-A*s].f[s], {s,0,t}])
```

The matrix exponential commands illustrated in the Section 5.6 application provide the basis for analogous *Maple* and MATLAB computations. You can then check routinely the answers for Problems 17 through 34 of this section.

8.3 Spectral Decomposition Methods

Here, we present an alternative approach to the computation of the matrix exponential $e^{\mathbf{A}t}$, one that does not require that eigenvectors (including generalized ones) of the $n \times n$ matrix $\mathbf{A}$ be found first. Assume that the characteristic polynomial of $\mathbf{A}$ is written in the form

$$p(\lambda) = (-1)^n|\mathbf{A} - \lambda\mathbf{I}|, \tag{1}$$

with leading term $+\lambda^n$. [Compare Eqs. (4) and (5) in Section 6.1.] If the (not necessarily distinct) eigenvalues of $\mathbf{A}$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n). \tag{2}$$