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1. Identify the type and obtain the solutions of the following first order equations:

(a) $(x^3 + \frac{y}{x})\,dx + (y^2 + \ln x)\,dy = 0$.

$M = x^3 + \frac{y}{x} \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x}$. $N = y^2 + \ln x \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{x}$ Thus, eqn is exact.

$F(x,y) = \int M\,dx + g(y) = \frac{1}{4}x^4 + y\ln x + g(y)$

where $g'(y) = N - \frac{\partial}{\partial y}\int M\,dx = y^2 + \ln|x| - \ln x = y^2$

$\Rightarrow g(y) = \frac{1}{3}y^3$

Soln $F(x,y) = C \Rightarrow \boxed{\frac{1}{4}x^4 + y\ln|x| + \frac{1}{3}y^3 = C}$

(b) $\frac{dy}{dx} = (2x + y - 1)^2$. linear coeffi.

Let $v = 2x + y - 1 \Rightarrow \frac{dv}{dx} = 2 + \frac{dy}{dx}$. eqn become

$\frac{dv}{dx} - 2 = v^2 \Rightarrow \frac{dv}{dx} = 2 + v^2 \Rightarrow \int \frac{dv}{2+v^2} = \int dx$

$\Rightarrow \frac{1}{\sqrt{2}}\tan^{-1}\frac{v}{\sqrt{2}} = x + C \Rightarrow \frac{v}{\sqrt{2}} = \tan(\sqrt{2}x + C_1) \Rightarrow 2x + y - 1 \cancel{= \sqrt{2}\tan(\sqrt{2}x+C}$

$= \sqrt{2}\tan(\sqrt{2}x + C_1)$

$\boxed{y = 1 - 2x + \sqrt{2}\tan(\sqrt{2}x + C_1)}$

(c) $x^2\frac{dy}{dx} = xy + y^2$. homogeneous

$\frac{dy}{dx} = \frac{xy + y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$. Let $v = \frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{dy}{dx}$

plug into the eqn. $v + x\frac{dv}{dx} = v + v^2 \Rightarrow x\frac{dv}{dx} = v^2 \Rightarrow \int\frac{dv}{v^2} = \int\frac{dx}{x}$

$\Rightarrow -\frac{1}{v} = \ln|x| + C \Rightarrow v = \frac{-1}{\ln|x| + C} \Rightarrow$

$\boxed{y = x \cdot v = \frac{-x}{1\ln|x| + C}}$

(d) $\frac{dy}{dx} = \frac{e^x}{y}$.   separable

$$\int y\,dy = \int e^x dx \Rightarrow \frac{1}{2}y^2 = e^x + c$$

$$\boxed{y = \pm\sqrt{2e^x + c_1}}$$

(e) $x\frac{dy}{dx} - 2y = x^3\cos x, \quad y(\pi/2) = 1.$ linear

$$P = -\frac{2}{x} \quad Q = x^2\cos x \Rightarrow \rho(x) = e^{\int P dx} = e^{-\int \frac{2}{x}dx}$$

$$= e^{-2\ln|x|} = x^{-2}$$

$$y = \frac{1}{\rho}\left(\int \rho \cdot Q\,dx + C\right) = x^2\left(\int x^{-2}\cdot x^2\cos x\,dx + C\right)$$

$$= x^2\left(\int \cos x\,dx + c\right)$$

$$\boxed{y = cx^2 + x^2\sin x}$$

$$1 = y\left(\frac{\pi}{2}\right) = c\frac{\pi^2}{4} + \frac{\pi^2}{4}$$

$$\boxed{c = \frac{4}{\pi^2} - 1}$$

(f) $x^2\frac{dy}{dx} + 2xy = 5y^3$.

Bernoulli $n = 3$, $\frac{dy}{dx} + \frac{2}{x}y = \frac{5}{x^2}y^3$

Let $v = y^{1-3} = y^{-2}$, plug into the eqn

$$\frac{dv}{dx} + (1-3)\frac{2}{x}v = (1-3)\frac{5}{x^2} \Rightarrow \text{linear} \quad \rho = e^{\int -\frac{4}{x}dx} = x^{-4}$$

$$v = x^4\left(\int x^{-4}\cdot\left(\frac{-10}{x^2}\right)dx + c\right) = x^4\left(2x^{-5} + c\right) = cx^4 + \frac{2}{x}$$

$$\boxed{y = \frac{1}{\pm\sqrt{v}} = \pm\frac{1}{\sqrt{cx^4 + \frac{2}{x}}}}$$

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2. Suppose the death rate of a population $P(t)$ at time t is $3P(t)$ and the birth rate is $2P(t)$. It is known that the initial population is 100. Set up a DE and solve it for $P(t)$.

$$\begin{cases} \dfrac{dP}{dt} = 2P - 3P = -P \\ P(0) = 100 \end{cases}$$

$$P(t) = 100e^{-t}$$

3. Solve $xy'' = y'$.

Let $v = y'$. eqn become $x\dfrac{dv}{dx} = v \Rightarrow \displaystyle\int \frac{dv}{v} = \int \frac{dx}{x}$

$\ln|v| = \ln|x| + C \Rightarrow v = C_1 x \Rightarrow \dfrac{dy}{dx} = C_1 x$

$$\boxed{y = \frac{C_1}{2}x^2 + C_2}$$

4. Solve $yy'' = 3(y')^2$.

Let $v = \dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\cdot\dfrac{dy}{dx} = v\cdot\dfrac{dv}{dy}$

Thus, eqn become

$$y\cdot v\cdot\frac{dv}{dy} = 3v^2 \Rightarrow \int \frac{dv}{v} = \int \frac{3}{y}\,dy$$

$$\Rightarrow \ln v = 3\ln y + C \Rightarrow v = C_1 y^3$$

Now $\dfrac{dy}{dx} = C_1 y^3 \Rightarrow \displaystyle\int \frac{dy}{y^3} = \int C_1\,dx$

$$\Rightarrow -\frac{1}{2y^2} = C_1 x + C_2 \rightarrow 2y^2 = \frac{-1}{C_1 x + C_2}$$

$$\boxed{y = \pm\sqrt{\frac{1}{C_3 x + C_4}} = \pm\frac{1}{\sqrt{C_3 x + C_4}}}$$

5. Given a DE $x\frac{dy}{dx} - 4x^2y + 2y\ln y = 0$. Use subtitution $v = \ln y$. Convert the equation to a new DE of v.

$$V = \ln y \;:\; \Rightarrow \frac{dV}{dx} = \frac{1}{y}\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y\frac{dV}{dx}$$

plug into the eqn

$$xy\frac{dv}{dx} - 4x^2y + 2yv = 0$$

divided by y, we obtain an eqn about V

$$x\frac{dv}{dx} - 4x^2 + 2V = 0$$

$$\boxed{\frac{dv}{dx} + \frac{2}{x}V = 4x}$$ linear.

6. The acceleration dv/dt of a Lamborghini is proportional to the differece between 250 km/h and the velocity v of the car. Set a DE for velocity v. DO NOT SOLVE !!!

$$\frac{dv}{dt} = k(250 - v)$$

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1. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices. Suppose $\det(\mathbf{A}) = 3$ and $\det(\mathbf{B}) = 5$. Are the following statements true or false?

   (a) $\det(\mathbf{AB}) = 15$. true

   (b) $\det(\mathbf{A} + \mathbf{B}) = 8$. false

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2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & -4 \\ 3 & 6 & 2 \end{bmatrix}.$$

   (a) Find $\mathbf{A}^{-1}$ by elementary row operation.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 7 & -4 & 0 & 1 & 0 \\ 3 & 6 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -3 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 18 & 1 & -7 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -38 & -2 & 15 \\ 0 & 1 & 0 & 18 & 1 & -7 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -38 & -2 & 15 \\ 18 & 1 & -7 \\ 3 & 0 & -1 \end{bmatrix}$$

   (b) Use cofactor expansion for first ~~rwo~~ row to find $det(\mathbf{A})$.

$$|A| = 1 \begin{vmatrix} 7 & -4 \\ 6 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & -4 \\ 3 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 7 \\ 3 & 6 \end{vmatrix}$$

$$= 38 - 36 + (-3) = -1$$

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3. Write out the definition of linear dependence of vectors $\mathbf{v_1}, \cdots, \mathbf{v_k}$. Then, show $\mathbf{v_1} = (2, 0, 1)$, $\mathbf{v_2} = (-3, 1, -1)$ and $\mathbf{v_3} = (0, -2, -1)$ are linearly dependent.

def: If $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ holds for some $c_1, \cdots, c_k$ not all zero, then $\vec{v}_1, \cdots, \vec{v}_k$ are linearly dep.

$$|\vec{v}_1, \vec{v}_2, \vec{v}_3| = \begin{vmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= 2 \cdot (-3) + 3 \cdot 2 = 0$$

Thus, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dep

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4. Write out the definition of a basis for a vector space V. Show $\mathbf{v_1} = (1, 2)^T$, $\mathbf{v_2} = (2, 1)^T$ is a basis for R^2.

If $\{\vec{v}_1, \vec{v}_2\}$ linearly indep, and $\text{span}\{\vec{v}_1, \vec{v}_2\} = R^2$ then $\{\vec{v}_1, \vec{v}_2\}$ is a basis of R^2

① $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \neq 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are linearly indep

② $\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$.

Pf: For any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, take $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

we get $c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

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5. Let $\mathbf{W}$ be a subset in $\mathbf{R}^4$ such that $x_1 = x_3$ and $x_2 = X_4$. Verify that $\mathbf{W}$ is a subspace of $\mathbf{R}^4$.

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_1 = x_3, \; x_2 = x_4 \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix} \right\}$$

Let $\vec{v}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, Then $\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + \tilde{x}_1 \\ x_2 + \tilde{x}_2 \\ x_1 + \tilde{x}_1 \\ x_2 + \tilde{x}_2 \end{pmatrix} \in W$

$$k\begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} kx_1 \\ kx_2 \\ kx_1 \\ kx_2 \end{pmatrix} \in W.$$

Therefore, W is a subspace of R^2.

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6. Find the general solution of

$$y'' - 5y' + 4y = 0.$$

$r^2 - 5r + 4 = 0 \Rightarrow r_1 = 1,\ r_2 = 4$

$$\boxed{y_h = c_1 e^{t} + c_2 e^{4t}}$$

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7. Find the solution of the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

$r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2 \Rightarrow y = c_1 e^{2t} + c_2 t e^{2t}$

$1 = y(0) = c_1 e^{0} + c_2 0 \cdot e^{0} = c_1$

$y' = 2c_1 e^{2t} + c_2 e^{2t} + c_2 2t e^{2t}$

$-1 = y'(0) = 2c_1 + c_2 \Rightarrow c_2 = -1 - 2c_1 = -3$

Thus

$$\boxed{y = e^{2t} - 3te^{2t}}$$

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8. Find the general solution of

$$y'' - 4y' + 5y = 0.$$

$r^2 - 4r + 5 = 0 \quad r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$y = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$

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9. Find the general solution of

$$y'' - 5y' + 4y = e^{2t}.$$

The homogeneous solution is from problem 6. Find y_p by undetermined coefficient method.

Let $y_p = Ae^{2t}$ plug into the eqn

$4Ae^{2t} - 5 \cdot 2Ae^{2t} + 4Ae^{2t} = e^{2t} \Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2}$

$\Rightarrow y_p = -\frac{1}{2}e^{2t}$. Then $\boxed{y = y_h + y_p = c_1 e^{t} + c_2 e^{4t} - \frac{1}{2}e^{2t}}$

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10. Use the variation parameter method to find the general solution of

$$y'' + 4y = 1$$

$y_h = C_1 \cos 2x + C_2 \sin 2x$ Since $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$y_p = u_1 \cos 2x + u_2 \sin 2x$. $W[\cos 2x, \sin 2x] = 2$ ~~plug into the~~ where

$u_1 = \int \frac{-1 \cdot \sin 2x}{W[y_1, y_2]} dx = -\frac{1}{2} \int \sin 2x \, dx = +\frac{1}{4} \cos 2x$,

$u_2 = \int \frac{1 \cdot \cos 2x}{W[y_1, y_2]} dx = \frac{1}{4} \sin 2x$.

Thus

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} \end{aligned}}$$

11. In problem 10, if we replace the nonhomogeneous term e^{2t} by e^{4t}, find y_p.

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$y_p = Ate^{4t}$. plug into the eqn

$(Ate^{4t})'' - 5(Ate^{4t})' + 4Ate^{4t} = e^{4t} \Rightarrow$ ~~$-5(Ae^{4t} + 4Ate^{4t})$~~

$A(4e^{4t} + 4e^{4t} + 16te^{4t}) - 5A(e^{4t} + 4te^{4t}) + 4Ate^{4t} = e^{4t}$

$(8A - 5A)e^{4t} = e^{4t}$

$\Rightarrow A = \frac{1}{3} \Rightarrow y_p = \frac{1}{3} te^{4t}$

$$\boxed{y = C_1 e^{t} + C_2 e^{4t} + \frac{1}{3} te^{4t}}$$

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12. Given an Euler equation

$$x^2 y'' + 2xy' - 6y = 0.$$

Let $y = x^r$ be solution. Plug it into the equation to derive a characteristic equation as we did for the constant coefficient equation. Solve the characteristic equation to get two root r_1, r_2. Then $y_1 = x^{r_1}, y_2 = x^{r_2}$ are two solutions to the Euler equation. Show they are linearly independent. Then write out the general solution.

$y = x^r$. plug into the eqn.

$x^2 (r)(r-1) x^{r-2} + 2x\, r \cdot x^{r-1} - 6x^r = 0$

$r(r-1) + 2r - 6 = 0 \Rightarrow r^2 + r - 6 = 0$

$r_1 = -3$, $r_2 = 2$, $y_1 = x^{-3}$. $y_2 = x^2$ are two solns

$W[y_1, y_2] = \begin{vmatrix} x^{-3} & x^2 \\ -3x^{-4} & 2x \end{vmatrix} = 2x^{-2} + 3x^{-2} = 5x^{-2} \neq 0. \Rightarrow y_1, y_2$ l. indep

$$\Rightarrow \boxed{y = C_1 x^{-3} + C_2 x^2}$$

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1. Use the Laplace Transform rules to find $F(s)$

   (a) $f(t) = e^{3t} - \sin 2t$ $\quad F(s) = \frac{1}{s-3} - \frac{2}{s^2+4}$

   (b) $f(t) = e^t \cos 2t$ $\quad F(s) = \frac{s-1}{(s-1)^2+4}$

   (c) $f(t) = t\sin t$ $\quad F(s) = -\frac{d}{ds}\frac{1}{s^2+1^2} = (s^2+1)^{-2}\cdot 2s = \frac{2s}{(s^2+1)^2}$

2. Use Laplace Transform to solve

$$y'' - 5y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

$$(s^2Y(s) - y(0)s) - 5(sY(s) - y(0)) + 4Y(s) = 0$$

$$(s^2 - 5s + 4)Y(s) = s - 5 \Rightarrow Y(s) = \frac{s-5}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4}$$

This leads to $\quad s - 5 = A(s-4) + B(s-1)$

let $s = 1 \Rightarrow -4 = -3A \Rightarrow A = \frac{4}{3}$

let $s = 4 \Rightarrow -1 = 3B \Rightarrow B = -\frac{1}{3}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{4}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\} - \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-4}\} \Rightarrow \boxed{y(t) = \frac{4}{3}e^t - \frac{1}{3}e^{4t}}$$

3. Use Laplace Transform to solve

$$y'' + 4y = 1, \qquad y(0) = 0, \quad y'(0) = 0.$$

$$s^2Y(s) + 4Y(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A(s^2+4) + s(Bs+C) = 1$$

let $s = 0 \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4}$

let $s = 1 \Rightarrow 5A + B + C = 1$

let $s = -1 \Rightarrow 5A + B - C = 1$

$\Rightarrow 2C = 0 \Rightarrow C = 0 \Rightarrow B = -\frac{1}{4}$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} \Rightarrow \boxed{y(t) = -\frac{1}{4}\cos 2t + \frac{1}{4}}$$

4. Use Laplace Transform to solve

$$y' + 4y = e^t, \qquad y(0) = 0, \quad \cancel{y'(0) = 0}.$$

$$sY(s) + 4Y(s) = \frac{1}{s-1} \quad \Rightarrow \quad Y(s) = \frac{1}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4}$$

$$A(s+4) + B(s-1) = 1$$

let $s = 1 \Rightarrow A = \frac{1}{5}$ $\quad \Longrightarrow \quad Y(s) = \frac{1}{5} \cdot \frac{1}{s-1} - \frac{1}{5}\, \frac{1}{s+4}$

let $s = -4 \Rightarrow B = -\frac{1}{5}$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}}$$

5. Find the general solution of the linear first order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}.$$

eigenvalue: $\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 2\lambda - 18 + 30 = \lambda^2 - 7\lambda + 12 = 0$

$$\boxed{\lambda_1 = 3 \qquad \lambda_2 = 4}$$

For $\lambda_1 = 3$, Solve $\qquad$ eigenvector

$$\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$$

For $\lambda_2 = 4$ solve

$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\vec{u}_1, \vec{u}_2$ linearly indep since $\lambda_1 \neq \lambda_2$

Thus, we have two linearly indep solns

$$\vec{X}_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t} \qquad \vec{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

general soln.

$$\boxed{\vec{X} = c_1 \begin{pmatrix} 5e^{3t} \\ -6e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix}}$$

6. In problem 5, add initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the constants c_1, c_2.

$$\vec{X}(0) = c_1 \begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{cases} 5c_1 + c_2 = 2 \\ -6c_1 - c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = -3 \\ c_2 = 17 \end{cases}$$

7. In problem 5, add a nonhomogeneous term $\mathbf{f}(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t$. Find a particular solution $\mathbf{x_p}(t)$ by the undetermined coefficient method.

Since $\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t$ is not in the homo soln in #5

the form of $\vec{X}_p(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$. Plug it into the eqn.

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{bmatrix} 9 & 5 \\ -6 & 2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t$$

$$\Rightarrow \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{cases} 8a_1 + 5a_2 = 3 \\ -6a_1 - 3a_2 = -1 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{2}{3} \\ a_2 = \frac{5}{3} \end{cases}$$

$$\vec{X}_p(t) = \cancel{\begin{pmatrix} \frac{4}{19} \\ \frac{8}{19} \end{pmatrix} e^t} \begin{pmatrix} -\frac{2}{3} \\ \frac{5}{3} \end{pmatrix} e^t$$

8. Find the general solution of the linear first order system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}.$$

$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 3\lambda - 9 + 18 = \lambda^2 + 9 = 0$$

$\lambda = \pm 3i$. For $\lambda = 3i$, we compute its complex eigenvector

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.,}$$

$9u_1 + (3-3i)u_2 = 0$. pick $u_2 = 3 \Rightarrow u_1 = -1+i$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1+i \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is ~~the~~ an eigenvector with

real part $\vec{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, imaginary part $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Thus

$$\vec{X}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cos 3t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 3t = \begin{pmatrix} -\cos 3t - \sin 3t \\ 3\cos t \end{pmatrix}$$

$$\vec{X}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 3t + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \sin 3t = \begin{pmatrix} \cos 3t - \sin 3t \\ 3\sin t \end{pmatrix}$$

general soln

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2$$

9. For the system in problem 5 with nonhomogeneous term given in problem 7, use variational parameter method to solve it.

$$\vec{f}(t) = \begin{pmatrix} -3e^{t} \\ e^{t} \end{pmatrix}$$

$$\Phi(t) = [\vec{x}_1(t), \vec{x}_2(t)] = \begin{bmatrix} 5e^{3t} & e^{4t} \\ -6e^{3t} & -e^{4t} \end{bmatrix} \quad \text{from \#5.}$$

$$\Phi^{-1}(t) = \frac{\begin{bmatrix} -e^{4t} & -e^{4t} \\ 6e^{3t} & 5e^{3t} \end{bmatrix}}{-5e^{7t} + 6e^{7t}} = \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ 6e^{-4t} & 5e^{-4t} \end{bmatrix}$$

$$\Phi^{-1}\vec{f} = \begin{pmatrix} -e^{-3t}\cdot(-3e^{t}) - e^{-3t}\cdot e^{t} \\ \cancel{-18}\ 6e^{-4t}(-3e^{t}) + 5e^{-4t}e^{t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -13e^{-3t} \end{pmatrix}$$

$$\int \Phi^{-1} f \, dt = \cancel{\begin{pmatrix} -\frac{1}{4}e^{2t} \\ -\frac{2}{3}e^{-3t} \end{pmatrix}} \begin{pmatrix} -e^{-2t} \\ \frac{13}{3}e^{-3t} \end{pmatrix}$$

$$\Phi \int \Phi^{-1} f \, dt = \begin{bmatrix} 5e^{3t} & e^{4t} \\ -6e^{3t} & -e^{4t} \end{bmatrix} \begin{pmatrix} -e^{-2t} \\ \frac{13}{3}e^{-3t} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}e^{t} \\ \frac{5}{3}e^{t} \end{pmatrix}$$