Math 3280 Exam I , II . III

Differential Equations and Linear Algebra

Dr Zhuangyi Liu

Summer, 2014

Name/Section______________________________________/__________ Score____________________

SHOW ALL WORK!

1. Identify the type and obtain the solutions of the following first order equations:

(a) $\frac{dy}{dx} = \frac{2x}{e^y}$

$$\int e^y dy = \int 2x\, dx$$

$$e^y = x^2 + C$$

$$y = \ln(x^2 + C)$$

(b) $\frac{dy}{dx} + \frac{1}{x}y = x, \quad y(1) = \frac{1}{3}.$ $\quad p(x) = \frac{1}{x} \quad Q(x) = x$

$$y = \frac{1}{\rho}\left(\int \rho \cdot Q\, dx + c\right) \quad \text{where } \rho(x) = e^{\int p(x)dx} = e^{\ln x} = x$$

$$= \frac{1}{x}\left(\int x^2 dx + c\right) = \frac{x^2}{3} + \frac{c}{x}$$

By $\frac{1}{3} = y(1) = \frac{1}{3} + \frac{c}{1} \Rightarrow c = 0$

Thus soln is $y = \frac{x^2}{3}$

(c) $x^2 \frac{dy}{dx} = x^2 + y^2 + xy.$

$$\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \qquad \text{Let } y = xv$$

$$\Rightarrow \quad x\frac{dv}{dx} + v = 1 + v^2 + v$$

$$\Rightarrow \quad \int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx \Rightarrow \tan^{-1} v = \ln|x| + c$$

$$y = x \tan(\ln|x| + c)$$

(d) $(2xy + \cos y)\,dx + (x^2 - x\sin y - 2y)\,dy = 0$

$M = 2xy + \cos y$. $N = x^2 - x\sin y - 2y$. $\frac{\partial M}{\partial y} = 2x - \sin y = \frac{\partial N}{\partial x}$

eqn is exact .

$\int M\,dx = x^2 y + x\cos y$. $g'(y) = N - \frac{\partial}{\partial y}\int M\,dx = -2y \Rightarrow g(y) = -y^2$

soln :

$$x^2 y + x\cos y - y^2 = C$$

(e) $\frac{dy}{dx} = \sin(x + y + 1) - 1$

Let $V = x + y + 1$. $\Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx}$

The eqn becomes $\frac{dV}{dx} - 1 = \sin V - 1 \Rightarrow \frac{dV}{dx} = \sin V$

$\Rightarrow \int \csc V\,dV = \int dx$

$\ln|\csc V - \cot V| = x + C$

$\ln|\csc(x+y+1) - \cot(x+y+1)| = x + C$

2. item Use substitution $v = 1/y$ to change $y' + 3xy = x^2y^2$ into a new equation about v. Point out the type of the new equation. Do not solve it.

Since $V = \frac{1}{y} \Rightarrow \frac{dV}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2\frac{dv}{dx} = -\frac{1}{V^2}\frac{dV}{dx}$. eqn becomes

$-\frac{1}{V^2}\frac{dV}{dx} + 3x\cdot\frac{1}{V} = x^2\cdot\frac{1}{V^2} \Rightarrow \boxed{\frac{dV}{dx} - 3xV = -x^2}$

3. Verify $y = e^x$ and e^{-3x} are solutions to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0.$$

~~$\frac{d}{dx} e^x = e^x$~~

① $(e^x)'' + 2(e^x)' - 3(e^x) = e^x + 2e^x - 3e^x = 0.$

② $(e^{-3x})'' + 2(e^{-3x})' - 3(e^{-3x}) = 9e^{-3x} - 6e^{-3x} - 3e^{-3x} = 0$

Thus $y = e^x$, and e^{-3x} are both solns to this eqn.

4. A tank contains 100 gallons of a solution dissolved salt and water, the mixture being kept uniform by stirring. If pure water is now allowed to flow into the tank at the rate of 2 gal/min, and the mixture flows out at the rate of 2 gal/min. There are 10 lb salt in the tank initially. Set up a DE with initial condition for find the amount of salt in the tank at time t. DON'T SOLVE.

Let $X(t)$ be the amount of salt at time t in the tank

$$\begin{cases} \frac{dx}{dt} = 0.2 - \frac{x}{100} \\ X(0) = 10 \text{ lb} \end{cases}$$

5. Suppose the death rate of a population $P(t)$ at time t is $3P(t)$ and the birth rate is $2P(t)$. It is known that the initial population is 100. Set up a DE for find $P(t)$. DON'T SOLVE.

$$\begin{cases} \frac{dP}{dt} = 2P - 3P \\ P(0) = 100 \end{cases}$$

6. Solve $xy'' = y'$.

Let $v = y' \Rightarrow v' = y''$

Thus, the eqn becomes $x \cdot v' = v \Rightarrow \frac{dv}{v} = \frac{dx}{x}$

$\ln|v| = \ln|x| + C_1$

$|v| = e^{C_1}|x| \Rightarrow v = C_2 x$, for $C_2 \neq 0$

Since $v = y'$, we have

$y = \frac{C_2}{2}x^2 + C_3$ for any C_2, C_3

1. Given the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -4 \\ 3 & 6 & -3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Use elementary row operations to reduce the augmented matrix $\boldsymbol{A}^{\#} = [\boldsymbol{A} \;\; \boldsymbol{b}]$ to Reduced Echelon matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 7 & -4 & 2 \\ 3 & 6 & -3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find all the solutions to the system.

$x_1 = -x_3 + 3$

$x_2 = x_3 - 1$

x_3, are free

or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad \forall x_3 \in \mathbb{R}$

2. In problem 1, replace the third entry of $\boldsymbol{b}$ by 1. Redo the problem.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 7 & -4 & 2 \\ 3 & 6 & -3 & \cancel{3}\,1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

inconsistent $\Longrightarrow$ no solution.

3. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}.$$

(a) Find A^{-1} by elementary row operation.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc} 1 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \Rightarrow \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

(b) Find $det(A)$.

$$|A| = 1 \cdot 2 - 2 \cdot 3 = -4$$

(c) Find the product matrix AA.

$$A \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$$

(d) Find the sum $A + A$.

$$A + A = \begin{bmatrix} 2 & 6 \\ 4 & 4 \end{bmatrix}$$

4. Find $\det(A)$ for A in (1) by cofactor expansion about row 2.

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 7 & -4 \\ 3 & 6 & -3 \end{vmatrix} = -3 \cdot \begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 7 \begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$$

5. Let $\boldsymbol{A}, \boldsymbol{B}$ be $n \times n$ matrices. Suppose $\det(\boldsymbol{A}) = 3$ and $\det(\boldsymbol{B}) = 5$. Are the following statements true or false?

(a) $\det(\boldsymbol{AB}) = 15$. true

(b) $\det(\boldsymbol{A}) + \det(\boldsymbol{B}) = 8$. ~~false~~ true

6. Show $\boldsymbol{v}_1 = (2, 0, 1)$, $\boldsymbol{v}_2 = (-3, 1, -1)$ and $\boldsymbol{v}_3 = (0, -2, -1)$ are linearly dependent by finding c_1, c_2, c_3 not all zero such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{0}.$$

The above eqn is

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\begin{vmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -3 & 0 \\ 1 & -2 \end{vmatrix} = -6 + 6 = 0$, we know there is infinitely many soln (c_1, c_2, c_3) for the above homogeneous system $\Rightarrow$ nonzero soln exists. Hence $\vec{v}_1, \vec{v}_2, \vec{v}_3$ l. dependent

7. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent, show $\boldsymbol{u}_1, \boldsymbol{u}_2$ are also linear independent where $\boldsymbol{u}_1 = \boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{u}_2 = \boldsymbol{v}_1 - \boldsymbol{v}_2$.

Let $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$. We show $c_1 = c_2 = 0$ is the only soln. Since $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 - \vec{v}_2$, we have

$$c_1(\vec{v}_1 + \vec{v}_2) + c_2(\vec{v}_1 - \vec{v}_2) = \vec{0}$$

i.e.,

$$(c_1 + c_2)\vec{v}_1 + (c_1 - c_2)\vec{v}_2 = \vec{0}.$$

But we are given that $\vec{v}_1, \vec{v}_2$ are linearly independent. By def.

$\begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$ is the only soln $\Rightarrow$ $\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$ is the only soln.

Done!

8. Show $v_1 = (1,2)$, $v_2 = (2,1)$ is a basis for R^2.

① $\vec{v}_1, \vec{v}_2$ are linearly indep. since $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$.

② Let $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$. then $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for

$\begin{cases} c_1 = -\frac{1}{3}x + \frac{2}{3}y \\ c_2 = \frac{2}{3}x - \frac{1}{3}y \end{cases}$ Thus. $\vec{v}_1, \vec{v}_2$ span R^2

By ①, ②, we've proved that $\vec{v}_1, \vec{v}_2$ is a basis for R^2

9. Find eigenvalues for matrix **A** in problem 3.

$$0 = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4$$

$$= (\lambda - 4)(\lambda + 1)$$

Thus, $\begin{cases} \lambda_1 = -1 \\ \lambda_2 = -4 \end{cases}$ are the eigenvalues of A.

10. For each eigenvalue in last problem, find a basis for it associated eigenspace.

① For $\lambda_1 = -1$. solve

$$\begin{pmatrix} 1-(-1) & 3 \\ 2 & 2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}y \\ y \end{pmatrix} = y \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

$E_{-1} = \operatorname{span}\left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\}$. its basis $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$

② For $\lambda_2 = 4$. Solve

$$\begin{pmatrix} 1-4 & 3 \\ 2 & 2-4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$E_4 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. its basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1. Find the general solution of

$$y'' - 5y' + 4y = 0.$$

$r^2 - 5r + 4 = 0 \Rightarrow r_1 = 1, \; r_2 = 4$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{4x}}$$

2. Find the solution of the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

$r^2 - 4r + 4 = 0 \Rightarrow \quad r_1 = r_2 = 2$

$y(x) = C_1 e^{2x} + C_2 x e^{2x}$. By initial condition $\begin{cases} 1 = y(0) = C_1 \\ -1 = y'(0) = 2C_1 + C_2 \end{cases}$

Thus. $C_1 = 1, \quad C_2 = -3$

$$\boxed{y(x) = e^{2x} - 3x e^{2x}}$$

3. Find the general solution of

$$y'' - 4y' + 5y = 0.$$

$r^2 - 4r + 5 = 0 \qquad r = \dfrac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm 2i$

$$\boxed{y(x) = C_1 e^{2x} \cos 2x + C_2 e^{2x} \sin 2x}$$

4. Find the general solution of

$$y'' - 5y' + 4y = e^{2t}.$$

The homogeneous solution is from problem 1.

Since $y_c(t) = c_1 e^{t} + c_2 e^{4t}$,

we pick $y_p = A e^{2t}$. plug into the eqn.

$4Ae^{2t} - 10Ae^{2t} + 4Ae^{2t} = e^{2t} \Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2}$

$$\boxed{y(t) = C_1 e^{t} + C_2 e^{4t} - \frac{1}{2} e^{2t}}$$

5. Use the variation parameter method to find the general solution of

$$y'' + 4y = 1$$

$y_c = C_1 \cos 2t + C_2 \sin 2t$. where $y_1 = \cos 2t$. $y_2 = \sin 2t$

$W[y_1, y_2] = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$

$u_1 = \int \frac{-f \cdot y_2}{W[y_1, y_2]} dt = \int \frac{-1 \cdot \sin 2t}{2} dt = \frac{\cos 2t}{4}$

$u_2 = \int \frac{f y_1}{W[y_1, y_2]} dt = \int \frac{1 \cdot \cos 2t}{2} dt = \frac{\sin 2t}{4} \Rightarrow y_p = u_1 y_1 + u_2 y_2 = \frac{1}{4}$

$\boxed{y(x) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4}}$

6. In problem 4, if we replace the nonhomogeneous term e^{2t} by e^{4t}, find y_p.

$y_p = Ate^{4t} \Rightarrow y_p' = Ae^{4t}(1+4t)$, $y_p'' = Ae^{4t}(8+16t)$

plug into the eqn

$8Ae^{4t} - 5Ae^{4t} = e^{4t} \Rightarrow A = \frac{1}{3}$

Thus. $\boxed{y_p = \frac{1}{3} t e^{4t}}$

7. Show $y_1 = 1$, $y_2 = x$, $y_3 = x^2$ are three linearly independent solutions to the differential equation $y''' = 0$.

Since $y_1''' = y_2''' = y_3''' = 0$, they are solns to $y''' = 0$

$W[y_1, y_2, y_3] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$

$\Rightarrow$ y_1, y_2, y_3 are linearly indep.

8. Find the general solution of the linear first order system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ where

$$\boldsymbol{A} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}.$$

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = (9-\lambda)(-2-\lambda) + 30 = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$$

$\lambda_1 = \cancel{4}\, 3, \quad \lambda_2 = 4.$

for $\lambda_1 = 3$. $\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$

for $\lambda_2 = 4$ $\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$

$$\vec{X} = C_1 \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

9. Find the general solution of the linear first order system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ where

$$\boldsymbol{A} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}.$$

$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda) + 18 = \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

for $\lambda_1 = 3i$

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_1 = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\vec{X}_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t = \begin{pmatrix} -2\cos 3t \\ 3\cos 3t - 3\sin 3t \end{pmatrix}$$

$$\vec{X}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t = \begin{pmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{pmatrix}$$

$$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2 = \begin{pmatrix} -2C_1\cos 3t - 2C_2\sin 3t \\ (3C_1 + 3C_2)\cos 3t + (-3C_1 + 3C_2)\sin 3t \end{pmatrix}$$

10. Convert the third order equation

$$y''' + 2y'' - 3y' + y = 0$$

into a first order system. Here the prime means derivative with respect to t.

Let $\begin{cases} x_1 = y \\ x_2 = y' \\ x_3 = y'' \end{cases} \Rightarrow$

$$\frac{dx_1}{dt} = y' = x_2$$

$$\frac{dx_2}{dt} = y'' = x_3$$

$$\frac{dx_3}{dt} = y''' = -y + 3y' - 2y'' = -x_1 + 3x_2 - 2x_3$$

Matrix form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$