Math 3298 Exam 1
NAME: **SCORE:**

18 pt 1. Given three points $A(1,0,1)$, $B(1,2,3)$, $C(0,-1,2)$.

(a) Find vectors $\vec{AB}$, $\vec{AC}$, $\vec{BC}$.

$$\vec{AB} = \langle 1-1, 2-0, 3-1 \rangle = \langle 0, 2, 2 \rangle$$
$$\vec{AC} = \langle 0-1, -1-0, 2-1 \rangle = \langle -1, -1, 1 \rangle$$
$$\vec{BC} = \langle 0-1, -1-2, 2-3 \rangle = \langle -1, -3, -1 \rangle$$

(b) Find $\vec{AB} + \vec{BC}$, $\vec{AB} - \vec{AC}$, and $2\vec{AB}$.

$$\vec{AB} + \vec{BC} = \vec{AC} = \langle -1, -1, 1 \rangle$$
$$\vec{AB} - \vec{AC} = \vec{CB} = -\vec{BC} = \langle 1, 3, 1 \rangle$$
$$2\vec{AB} = \langle 0, 4, 4 \rangle$$

(c) Find length of $\vec{AB}$ and area of the parallelogram spanned by $\vec{AB}$, $\vec{AC}$.

$$|\vec{AB}| = \sqrt{0 + 2^2 + 2^2} = \sqrt{8}$$
$$\text{area} = |\vec{AB} \times \vec{AC}| = |\langle 4, 2, 2 \rangle| = \sqrt{16+4+4} = \sqrt{24}$$

6 pt 2. Given vectors $\mathbf{a}, \mathbf{b}$. Find $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}$.

Since $(\vec{a} \times \vec{b}) \perp \vec{a}$, we have $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

6 pt 3. Given two unit vectors $\mathbf{u}, \mathbf{v}$. The angle between them is $\pi/4$. What is $\mathbf{u} \cdot \mathbf{v}$?

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\frac{\pi}{4} = 1 \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

(8 pt) 4. Find the equation of a line passing point $P(1,1,5)$ and parallel to another line whose equation is $x = 1+t,\ y = 3-5t,\ z = 9+2t$.

direction $\vec{v}$ of the given line is $\langle 1, -5, 2\rangle$.

The line passing $P(1,1,5)$ with direction $\langle 1, -5, 2\rangle$ is

$$\begin{cases} x = 1+t \\ y = 1-5t \\ z = 5+2t \end{cases}$$

(8 pt) 5. Find equation of the plane which pass through point $P(2,1,1)$ and parallel to another plane $x+2y-z=0$.

The normal dir of the given plane is $\vec{n} = \langle 1, 2, -1\rangle$

then it is also the normal dir of the plane we are looking for. Since it passes $P(2,1,1)$, its eqn is

$$\boxed{(x-2)+2(y-1)-(z-1) = 0.}$$

(5 pt) 6. Repeat above problem after changing "parallel" to "perpendicular". We ~~also~~ are given that ~~for~~ the plane also pass $Q(1,0,1)$

$\vec{PQ} = \langle -1, -1, 0\rangle$ // to the plane

the normal dir of the plane $x+2y-z=0$ is $\vec{n_1} = \langle 1, 2, -1\rangle$ also // to the plane

Then, $\vec{n} = \vec{PQ} \times \vec{n_1} = \langle 1, -1, -1\rangle$ is $\perp$ to the plane $\Rightarrow$ normal dir.

plane eqn: $\boxed{(x-2)-(y-1)-(z-1)=0}$

(7 pt) 7. Find the point in which the line $x = 2-t,\ y = 1+3t,\ z = 4t$ intersects the plane $2x-y+z=2$.

At intersection point. $2(2-t)-(1+3t)+4t = 2$

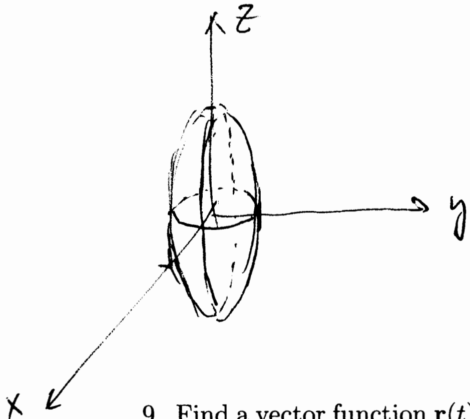
$\Rightarrow -t = -1 \Rightarrow t = 1$.

Then. $\begin{cases} x = 2-1 \\ y = 1+3\cdot 1 \\ z = 4\cdot 1 \end{cases} \Rightarrow (1, 4, 4)$

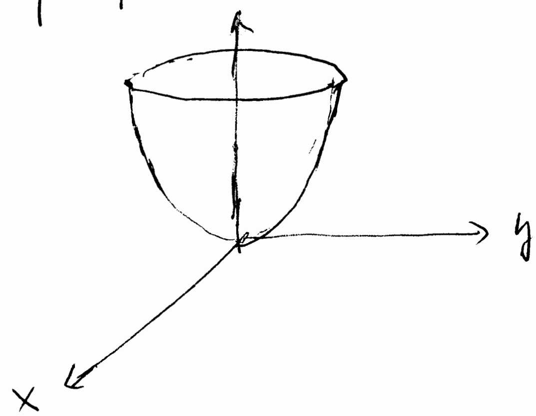
7pt

8. Make a rough sketch of the quadratic surfaces, and give their names (i) $4x^2 + 9y^2 + z^2 = 1$. (ii) $x^2 + 4y^2 - z = 0$.

ellipsoid



elliptic paraboloid



7pt

9. Find a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for the curve which is the intersection of surfaces $4 = x^2 + y^2$ and $z^2 = x^2 + y^2$.

Let $x = 2\cos t$, $y = 2\sin t \Rightarrow x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4$

$\Rightarrow z^2 = x^2 + y^2 = 4 \Rightarrow z = \pm 2$

$\vec{r}(t) = \langle 2\cos t, 2\sin t, \pm 2 \rangle \qquad 0 \le t \le 2\pi$

7pt

10. Suppose $\mathbf{a} \neq \mathbf{0}$. If $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, can we claim that $\mathbf{b} = \mathbf{c}$? If Yes, prove it. If no, give an example to support your conclusion.

No! example: $\vec{a} = \vec{i} \quad \vec{b} = 3\vec{i}. \quad \vec{c} = 2\vec{i}$

$\vec{a} \times (\vec{b} - \vec{c}) = \vec{i} \times \vec{i} = \vec{0}$

and $\vec{b} \neq \vec{c}$.

7 pt 11. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Find its velocity, speed, and acceleration.

velocity $\vec{V}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$

speed $v(t) = |\vec{V}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

acceleration $\vec{a}(t) = \vec{r}\,''(t) = \langle -\cos t, -\sin t, 0\rangle$

7 pt 12. In above problem, find the tangential and normal components of the acceleration. Also, find $\mathbf{T}(t), \mathbf{N}(t)$

$a_T = v'(t) = 0$, $a_N = K \cdot v^2 = K \cdot (\sqrt{2})^2 = 2K = 2 \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

$= \frac{2|\langle \sin t, -\cos t, 1\rangle|}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \Rightarrow \vec{T}\,'(t) = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle$. $|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}$

$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \langle -\cos t, -\sin t, 0\rangle$, clearly $\vec{a}(t) = \vec{r}\,''(t) = 0 \cdot \vec{T}(t) + 1 \cdot \vec{N}(t)$

$= \vec{N}(t)$

7 pt 13. A particle starts at the origin with initial velocity $\mathbf{v}(0) = \langle 0, 0, 0\rangle$. Its acceleration is $\mathbf{a}(t) = \langle 2, 1, 0\rangle$. Find the distance it travels from $t = 0$ to $t = 1$.

$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle 2t, t, 0\rangle + \langle C_1, C_2, C_3\rangle$

since $\vec{V}(0) = \langle 0, 0, 0\rangle \Rightarrow \langle C_1, C_2, C_3\rangle = \langle 0, 0, 0\rangle$

Thus. $\vec{v}(t) = \langle 2t, t, 0\rangle \Rightarrow |\vec{v}(t)| = \sqrt{(2t)^2 + t^2} = \sqrt{5}\,t$

travel distance = curve length = $\int_0^1 \sqrt{5}\,t\,dt = \frac{\sqrt{5}}{2}$