Math 3298 Exam II, Part I
NAME: **SCORE:**

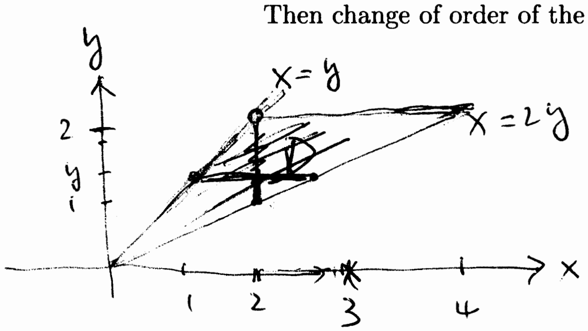
1. Evaluate the iterated integral

$$\int_0^2 \int_y^{2y} xy\,dx\,dy.$$

$$= \int_0^2 y \cdot \frac{x^2}{2}\Big|_y^{2y} dy = \int_0^2 \frac{y}{2}(4y^2 - y^2)\,dy$$

$$= \int_0^2 \frac{3}{2} y^3\,dy = \frac{3}{8} y^4\Big|_0^2 = \frac{3}{8}\cdot 16 = 6$$

2. Sketch the region D of the double integral associated with the iterated integral in last problem. Then change of order of the iterated integral to $dydx$. Evaluate it.



$$\int_0^2 \int_{\frac{x}{2}}^{x} xy\,dy\,dx + \int_2^4 \int_{\frac{x}{2}}^{2} xy\,dy\,dx$$

$$= \int_0^2 \frac{x}{2} y^2\Big|_{\frac{x}{2}}^{x} dx + \int_2^4 \frac{x}{2} y^2\Big|_{\frac{x}{2}}^{2} dx$$

$$= \int_0^2 \frac{3}{8} x^3\,dx + \int_2^4 \left(2x - \frac{x^3}{8}\right) dx$$

$$= \frac{3}{32}\cdot x^4\Big|_0^2 + \left(x^2 - \frac{x^4}{32}\right)\Big|_2^4 = \frac{3}{2} + (16-8) - \frac{7}{2} = 6$$

3. Find the volume of the solid under the surface $z = 4 - x^2 - y^2$, and over the rectangle $D = [-1,1] \times [-1,1]$.

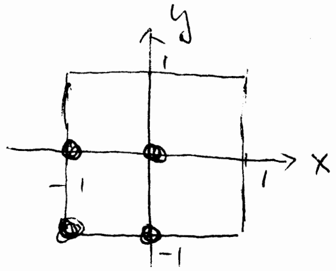
$$V = \int_{-1}^1 \int_{-1}^1 (4 - x^2 - y^2)\,dx\,dy = \int_{-1}^1 \left(4x - \frac{x^3}{3} - y^2 x\right)\Big|_{-1}^1 dy$$

$$= \int_{-1}^1 \left(8 - \frac{2}{3} - 2y^2\right) dy = \left(\frac{22}{3} y - \frac{2}{3} y^3\right)\Big|_{-1}^1 = 2\left(\frac{22}{3} - \frac{2}{3}\right)$$

$$= \frac{40}{3}$$

4. Divide D in above problem into four sub-rectangles equally ($m = n = 2$). Take the sample point as the low left corner point of each sub-rectangle. Compute the Riemann sum.

$\Delta A = 1$



$$\left[f(-1,-1) + f(0,-1) + f(0,0) + f(-1,0) \right] \Delta A$$

$$= (4-1-1) + (4-0-1) + (4-0-0) + (4-1-0)$$

$$= 2 + 3 + 4 + 3 = 12$$

5. Find the local max and min values and saddle points of the function $f(x, y) = x^2 + xy + y^2 + y$.

$$\begin{cases} f_x = 2x + y = 0 \\ f_y = x + 2y + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3} \\ y = -\frac{2}{3} \end{cases} \quad \text{c.p } \left(\tfrac{1}{3}, -\tfrac{2}{3}\right)$$

$f_{xx} = 2 > 0 \quad f_{xy} = 1 = f_{yx} \quad f_{yy} = 2$

$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} > 0$, By 2nd derivative test

$f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$ is the local min.

6. Use Lagrange multiplier method to find the minimum of $x^2 + y^2 + z^2$ where x, y, z are positive numbers and $x + y + z = 12$.

$\min f(x, y, z) = x^2 + y^2 + z^2$

s.t. $x + y + z = 12 \dashrightarrow g(x, y, z) = x + y + z - 12$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases}$$

Since $x + y + z = 12$, we have

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 12 \Rightarrow \lambda = 8$$

Then $x = y = z = 4$

minimum is $f(4,4,4) = 16 + 16 + 16 = 48$