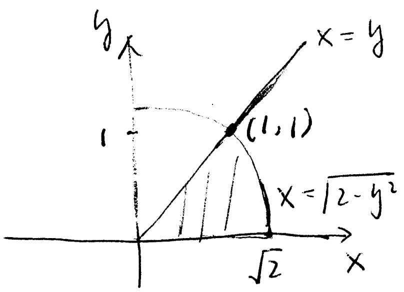
Math 3298 Exam II, Part II

NAME: **SCORE:**

1. Calculate the iterated integral $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$ by converting it to polar coordinates. Also sketch the region of integration.



$$\int_0^{\sqrt{2}} \int_0^{\frac{\pi}{4}} (r\cos\theta + r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^{\sqrt{2}} r^2\, dr \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{\sqrt{8}}{3} \cdot (\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{4}} = \frac{1}{3}\left(\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - (0-1)\right)$$

$$= \frac{\sqrt{8}}{3}$$

2. Find the area of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x.$ $\qquad D = \{(x,y) \mid x^2 + y^2 \le 1\}$

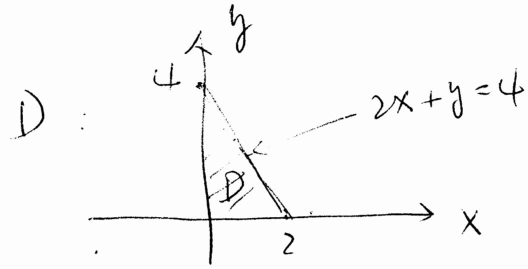
$$\iint_D \sqrt{1 + y^2 + x^2}\, dA$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+r^2}\; r\, dr\, d\theta = 2\pi\; \frac{2}{3}(1+r^2)^{\frac{3}{2}} \cdot \frac{1}{2}\Big|_0^1$$

$$= \frac{2\pi}{3}\left(\cancel{\sqrt{2}^3}\; 2^{\frac{3}{2}} - 1\right)$$

3. Set up and calculate a triple integral to find volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.

$$\iiint_E dV = \iint_D \left(\int_0^{4-2x-y} dz \right) dA$$



$$= \iint_D (4-2x-y)\, dA$$

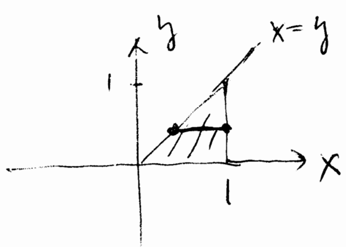
$$= \int_0^2 \int_0^{4-2x} (4-2x-y)\, dy\, dx = \int_0^2 \left[(4-2x)y - \frac{1}{2}y^2 \right] \Big|_0^{4-2x} dx$$

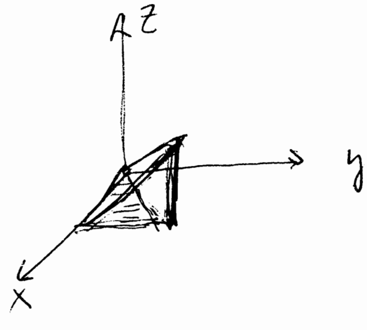
$$= \int_0^2 \frac{1}{2}(4-2x)^2 dx = \frac{1}{2} \cdot \frac{1}{3}(4-2x)^3\left(-\frac{1}{2}\right) \Big|_0^2 = \frac{1}{12} \cdot 4^3 = \frac{16}{3}$$

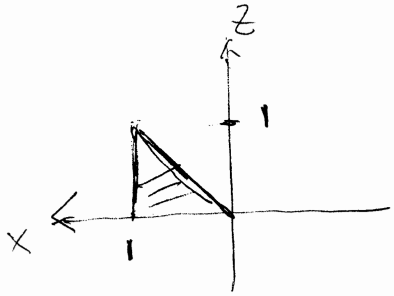
4. Change the integration order of the following iterated integral

$$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$$

to $dydzdx$.







$$\int_0^1 \int_0^x \int_z^x f(x, y, z)\, dy\, dz\, dx$$

5. Find the volume of the part of the ball $\rho \le a$ that lies between $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$

$$V = \iiint_E dV = \int_0^a \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \frac{1}{3}\rho^3 \Big|_0^a \cdot \theta \Big|_0^{2\pi} \cdot (-\cos\phi) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{3} a^3 \cdot 2\pi \cdot \left(-\cos\frac{\pi}{3} + \cos\frac{\pi}{6}\right)$$

$$= \frac{2\pi}{3} a^3 \cdot \left(+\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{(-1+\sqrt{3})}{3} \pi a^3$$

6. (bonus)

Sketch the region in last problem.