Math 3298 Exam III, part I

NAME: SCORE:

1. Use the transformation (in another word, changing variables) $x = u^2, y = v^2, z = w^2$ to find the volume of the region bounded by the three coordinate planes and the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$$

Let E be the region in xyz space. S be the region in uvw space

$$V = \iiint_E dV = \iiint_S \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$= 8\int_0^1\int_0^{1-u}\int_0^{1-u-v} uvw\,dw\,dv\,du = 4\int_0^1\int_0^{1-u} uv(1-u-v)^2\,dv\,du$$

$$= 4\int_0^1\int_0^{1-u}[u(1-u)^2v - 2u(1-u)\cdot v^2 + uv^3]\,dv\,du = \int_0^1 \frac{1}{3}u(1-u)^4\,du = \frac{1}{90}$$

2. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where the curve C is the arc of parabola $x = 1 - y^2$ from $(0,-1)$ to $(0,1)$ and $\mathbf{F} = \langle y^3, x^2\rangle$.

Let $y = t \Rightarrow x = 1 - t^2$. $-1 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C y^3\,dx + x^2\,dy$$

$$= \int_{-1}^1 t^3(-2t)\,dt + (1-t^2)^2\,dt$$

$$= \int_{-1}^1 (-2t^4 + 1 - 2t^2 + t^4)\,dt$$

$$= \left(-\frac{1}{5}t^5 - \frac{2}{3}t^3 + t\right)\Big|_{-1}^{1} = \frac{4}{15}$$

3. Verify that the vector field $\mathbf{F} = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3 \rangle$ is conservative. Then find a function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

$$P = 4x^3y^2 - 2xy^3 \qquad Q = 2x^4y - 3x^2y^2 + 4y^3$$

$$\frac{\partial P}{\partial y} = 8x^3y - 6xy^2 = \frac{\partial Q}{\partial x} = 8x^3y - 6xy^2 \Rightarrow \text{conservative}$$

$$f_x = 4x^3y^2 - 2xy^3 \Rightarrow f(x,y) = x^4y^2 - x^2y^3 + g(y)$$

$$\Downarrow$$

$$f_y = 2x^4y - 3x^2y^2 + 4y^3 \quad ②$$

$$f_y = 2x^4y - 3x^2y^2 + g'(y) \quad ③$$

from ②, ③ we get $g'(y) = 4y^3 \Rightarrow g(y) = y^4$

Hence $\boxed{f(x,y) = x^4y^2 - x^2y^3 + y^4}$

4. Evaluate the line integral

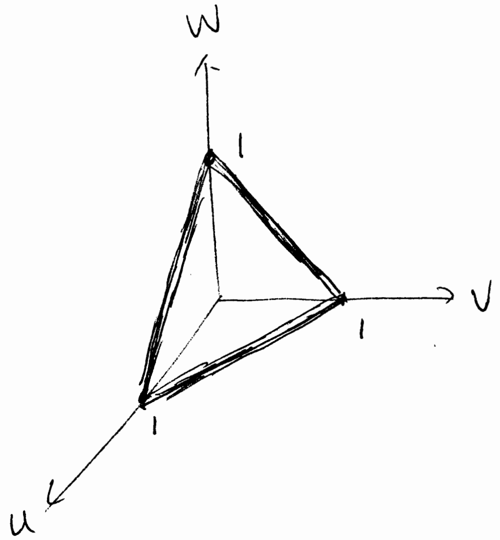
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

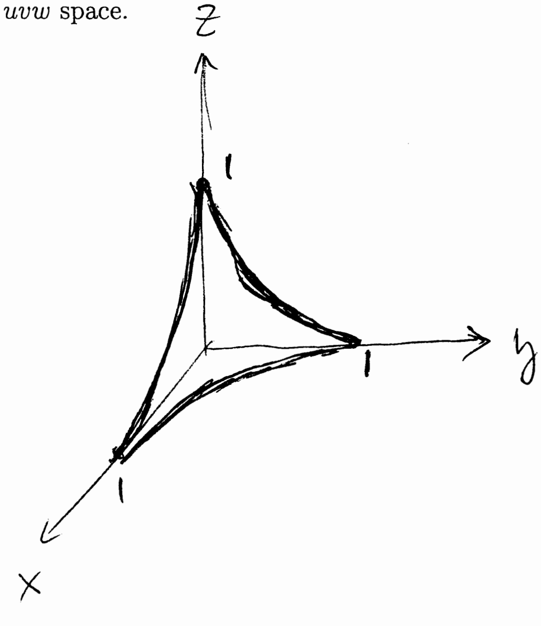
where $\mathbf{F}$ is given in last problem. $C: \mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t \rangle$ for $0 \le t \le 1$. (Hint: use the fact that $\mathbf{F}$ is conservative).

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,1) - f(0,1) = 1 - 1 = 0$$

5. Sketch the region in problem 1 in xyz space and in uvw space.





Math 3298 Exam III, part II
NAME: **SCORE:**

1. Use the Green's Theorem to evaluate the linear integral $\int_C -x^2y\,dx + xy^2\,dy$, where C is the circle $x^2+y^2=4$ with counterclockwise orientation.

$P = -x^2y, \quad Q = xy^2 \qquad D = \{(x,y) \mid x^2+y^2 \le 4\}$

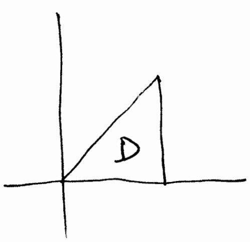
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - (-x^2) = x^2+y^2$$

$$\int_C -x^2y\,dx + xy^2\,dy = \iint_D (x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = 2\pi \cdot \frac{1}{4}r^4 \Big|_0^2$$

$$= 8\pi$$

2. Find the area of the part of the surface $z = \frac{x^2}{2}$ that lies above the triangle with vertices $(0,0),(1,0),(1,1)$.



$$A(S) = \iint_S 1\,dS = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_D \sqrt{1+x^2+0^2}\,dA = \int_0^1\int_0^x \sqrt{1+x^2}\,dy\,dx$$

$$= \int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{3}\left(2^{\frac{3}{2}}-1\right)$$

3. Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2,\ xy,\ z\rangle$ and S is the part of paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\cdot\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \qquad D = \{(x,y) \mid x^2+y^2 \le 1\}$$

$$= \iint_D \left[(-2x)x^2 + (-2y)xy + (x^2+y^2)\right] dA$$

$$= \iint_D \left[-2x(x^2+y^2) + (x^2+y^2)\right] dA$$

$$= \int_0^{2\pi}\int_0^1 (-2r^3\cos\theta + r^2)\, r\,dr\,d\theta$$

$$= 0 + \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta \qquad \text{since } \int_0^{2\pi}\cos\theta\,d\theta = 0$$

$$= \frac{\pi}{2}$$

4. Given $\mathbf{F} = \langle e^{-x}\sin y,\ e^{-y}\sin z,\ e^{-z}\sin x\rangle$. Calculate curl$\mathbf{F}$ and div$\mathbf{F}$.

$$\text{Curl }\vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ -\frac{\partial P}{\partial z} + \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

$$= \langle 0 - e^{-y}\cos z,\ -e^{-z}\cos x,\ -e^{-x}\cos y\rangle$$

$$\text{div }\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$$

5. In last problem, show div(curl$\mathbf{F}$) $= 0$.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \operatorname{div}\langle -e^{-y}\cos z, \; -e^{-z}\cos x, \; -e^{-x}\cos y\rangle$$

$$= 0 + 0 + 0 = 0$$