

Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph.

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for \mathbf{F} .

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field \mathbf{F} in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

16.1 Exercises

10 Sketch the vector field \mathbf{F} by drawing a diagram like Figure 5 or Figure 9.

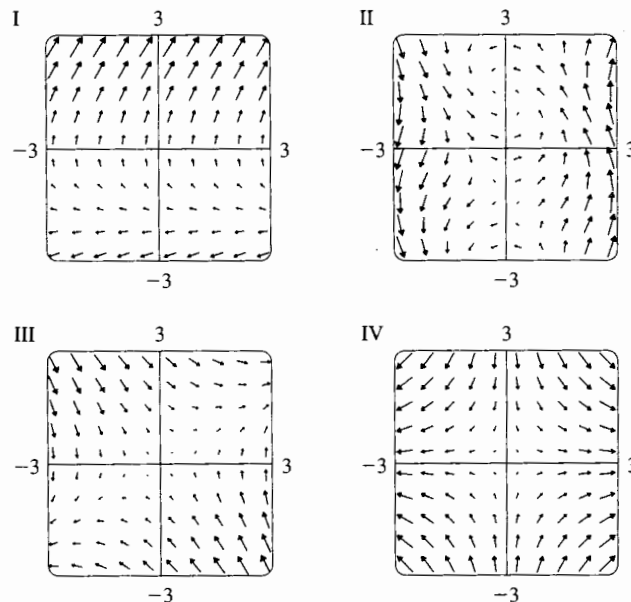
1. $\mathbf{F}(x, y) = 0.3 \mathbf{i} - 0.4 \mathbf{j}$
2. $\mathbf{F}(x, y) = \frac{1}{2}x \mathbf{i} + y \mathbf{j}$
3. $\mathbf{F}(x, y) = -\frac{1}{2} \mathbf{i} + (y - x) \mathbf{j}$
4. $\mathbf{F}(x, y) = y \mathbf{i} + (x + y) \mathbf{j}$
5. $\mathbf{F}(x, y) = \frac{y \mathbf{i} + x \mathbf{j}}{\sqrt{x^2 + y^2}}$
6. $\mathbf{F}(x, y) = \frac{y \mathbf{i} - x \mathbf{j}}{\sqrt{x^2 + y^2}}$
7. $\mathbf{F}(x, y, z) = \mathbf{k}$
8. $\mathbf{F}(x, y, z) = -y \mathbf{k}$
9. $\mathbf{F}(x, y, z) = x \mathbf{k}$
10. $\mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$

11–14 Match the vector fields \mathbf{F} with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle x, -y \rangle$
12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$

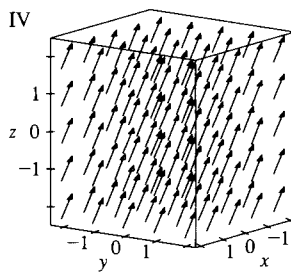
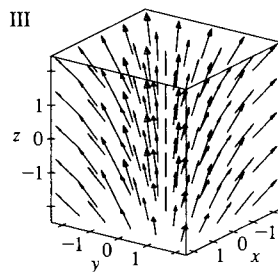
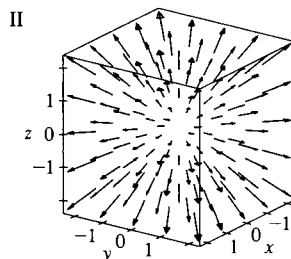
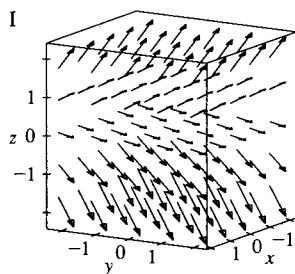


15–18 Match the vector fields \mathbf{F} on \mathbb{R}^3 with the plots labeled I–IV. Give reasons for your choices.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ 16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$

17. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$

18. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



CAS 19. If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

CAS 20. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

21–24 Find the gradient vector field of f .

21. $f(x, y) = xe^{xy}$

22. $f(x, y) = \tan(3x - 4y)$

23. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

24. $f(x, y, z) = x \ln(y - 2z)$

25–26 Find the gradient vector field ∇f of f and sketch it.

25. $f(x, y) = x^2 - y$

26. $f(x, y) = \sqrt{x^2 + y^2}$

CAS 27–28 Plot the gradient vector field of f together with a contour map of f . Explain how they are related to each other.

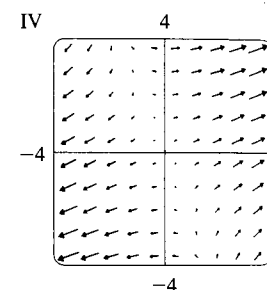
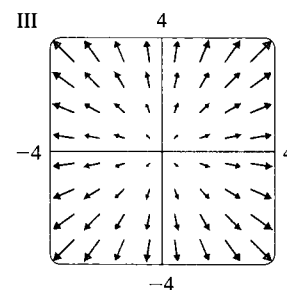
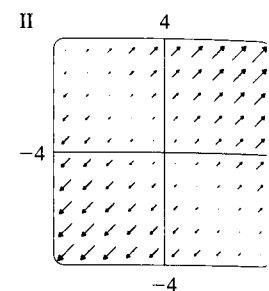
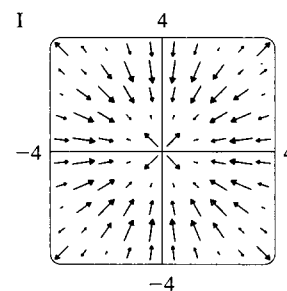
29–32 Match the functions f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

29. $f(x, y) = x^2 + y^2$

30. $f(x, y) = x(x + y)$

31. $f(x, y) = (x + y)^2$

32. $f(x, y) = \sin \sqrt{x^2 + y^2}$



33. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

34. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

35. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

36. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

Thus

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (t^3 + 5t^6) dt = \left[\frac{t^4}{4} + \frac{5t^7}{7} \right]_0^1 = \frac{27}{28}\end{aligned}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field \mathbf{F} on \mathbb{R}^3 is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. We use Definition 13 to compute its line integral along C :

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt\end{aligned}$$

But this last integral is precisely the line integral in [10]. Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

For example, the integral $\int_C y dx + z dy + x dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

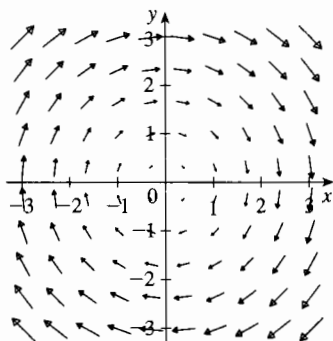
$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

16.2 Exercises

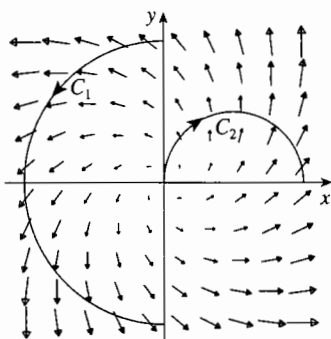
1–16 Evaluate the line integral, where C is the given curve.

- $\int_C y^3 ds$, $C: x = t^3$, $y = t$, $0 \leq t \leq 2$
- $\int_C xy ds$, $C: x = t^2$, $y = 2t$, $0 \leq t \leq 1$
- $\int_C xy^4 ds$, C is the right half of the circle $x^2 + y^2 = 16$
- $\int_C x \sin y ds$, C is the line segment from $(0, 3)$ to $(4, 6)$
- $\int_C (x^2 y^3 - \sqrt{x}) dy$,
 C is the arc of the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$
- $\int_C e^x dx$,
 C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$
- $\int_C (x + 2y) dx + x^2 dy$, C consists of line segments from $(0, 0)$ to $(2, 1)$ and from $(2, 1)$ to $(3, 0)$
- $\int_C x^2 dx + y^2 dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment from $(0, 2)$ to $(4, 3)$
- $\int_C xyz ds$,
 $C: x = 2 \sin t$, $y = t$, $z = -2 \cos t$, $0 \leq t \leq \pi$
- $\int_C xyz^2 ds$,
 C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$
- $\int_C xe^{yz} ds$,
 C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$
- $\int_C (x^2 + y^2 + z^2) ds$,
 $C: x = t$, $y = \cos 2t$, $z = \sin 2t$, $0 \leq t \leq 2\pi$
- $\int_C xye^{yz} dy$, $C: x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 1$
- $\int_C y dx + z dy + x dz$,
 $C: x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \leq t \leq 4$
- $\int_C z^2 dx + x^2 dy + y^2 dz$, C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$
- $\int_C (y + z) dx + (x + z) dy + (x + y) dz$, C consists of line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$

17. Let \mathbf{F} be the vector field shown in the figure.
- If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.
 - If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



18. The figure shows a vector field \mathbf{F} and two curves C_1 and C_2 . Are the line integrals of \mathbf{F} over C_1 and C_2 positive, negative, or zero? Explain.



19–22 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

- $\mathbf{F}(x, y) = xy\mathbf{i} + 3y^2\mathbf{j}$,
 $\mathbf{r}(t) = 11t^4\mathbf{i} + t^3\mathbf{j}$, $0 \leq t \leq 1$
- $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y - z)\mathbf{j} + z^2\mathbf{k}$,
 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$, $0 \leq t \leq 1$
- $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$,
 $\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$
- $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$,
 $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi$

23–26 Use a calculator or CAS to evaluate the line integral correct to four decimal places.

- $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\mathbf{i} + \sin y\mathbf{j}$ and $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t^2}\mathbf{j}$, $1 \leq t \leq 2$

- $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = y \sin z\mathbf{i} + z \sin x\mathbf{j} + x \sin y\mathbf{k}$ and $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$, $0 \leq t \leq \pi$
- $\int_C x \sin(y + z) ds$, where C has parametric equations $x = t^2$, $y = t^3$, $z = t^4$, $0 \leq t \leq 5$
- $\int_C ze^{-xy} ds$, where C has parametric equations $x = t$, $y = t^2$, $z = e^{-t}$, $0 \leq t \leq 1$

CAS 27–28 Use a graph of the vector field \mathbf{F} and the curve C to guess whether the line integral of \mathbf{F} over C is positive, negative, or zero. Then evaluate the line integral.

- $\mathbf{F}(x, y) = (x - y)\mathbf{i} + xy\mathbf{j}$,
 C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

- $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$,
 C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

- (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\mathbf{i} + xy\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $0 \leq t \leq 1$.



- (b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0, 1/\sqrt{2}$, and 1 (as in Figure 13).

- (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k}$, $-1 \leq t \leq 1$.



- (b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm \frac{1}{2}$ (as in Figure 13).

CAS 31. Find the exact value of $\int_C x^3 y^2 z ds$, where C is the curve with parametric equations $x = e^{-t} \cos 4t$, $y = e^{-t} \sin 4t$, $z = e^{-t}$, $0 \leq t \leq 2\pi$.

- (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counter-clockwise direction.

CAS

- (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

- A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geq 0$. If the linear density is a constant k , find the mass and center of mass of the wire.

- A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a . If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

- (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

- (b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \leq t \leq 2\pi$, if the density is a constant k .

36. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq 2\pi$, if the density at any point is equal to the square of the distance from the origin.
37. If a wire with linear density $\rho(x, y)$ lies along a plane curve C , its **moments of inertia** about the x - and y -axes are defined as

$$I_x = \int_C y^2 \rho(x, y) \, ds \quad I_y = \int_C x^2 \rho(x, y) \, ds$$

Find the moments of inertia for the wire in Example 3.

38. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C , its **moments of inertia** about the x -, y -, and z -axes are defined as

$$I_x = \int_C (y^2 + z^2) \rho(x, y, z) \, ds$$

$$I_y = \int_C (x^2 + z^2) \rho(x, y, z) \, ds$$

$$I_z = \int_C (x^2 + y^2) \rho(x, y, z) \, ds$$

Find the moments of inertia for the wire in Exercise 35.

39. Find the work done by the force field $\mathbf{F}(x, y) = x \mathbf{i} + (y + 2) \mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j}$, $0 \leq t \leq 2\pi$.
40. Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} + ye^x \mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.
41. Find the work done by the force field $\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$ on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.
42. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 5 in Section 16.1.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.
43. The position of an object with mass m at time t is $\mathbf{r}(t) = at^2 \mathbf{i} + bt^3 \mathbf{j}$, $0 \leq t \leq 1$.
- (a) What is the force acting on the object at time t ?
- (b) What is the work done by the force during the time interval $0 \leq t \leq 1$?
44. An object with mass m moves with position function $\mathbf{r}(t) = a \sin t \mathbf{i} + b \cos t \mathbf{j} + ct \mathbf{k}$, $0 \leq t \leq \pi/2$. Find the work done on the object during this time period.

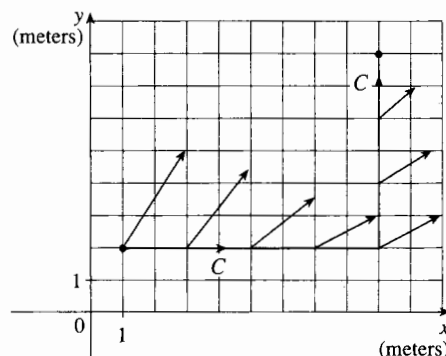
45. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?
46. Suppose there is a hole in the can of paint in Exercise 45 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?
47. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
- (b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?
48. The base of a circular fence with radius 10 m is given by $x = 10 \cos t$, $y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2 . Sketch the fence and determine how much paint you will need if you paint both sides of the fence.
49. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, and \mathbf{v} is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} [|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2]$$

51. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field \mathbf{F} are measured in newtons by the scales on the axes. Estimate the work done by \mathbf{F} on the object.



52. Experiments show that a steady current I in a long wire produces a magnetic field \mathbf{B} that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric

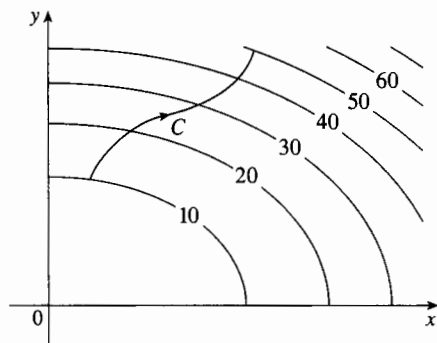
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot dr$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot dr$, where C has parametric equations

$$x = t^2 + 1 \quad y = t^3 + t \quad 0 \leq t \leq 1$$

$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

- 3–10 Determine whether or not \mathbf{F} is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

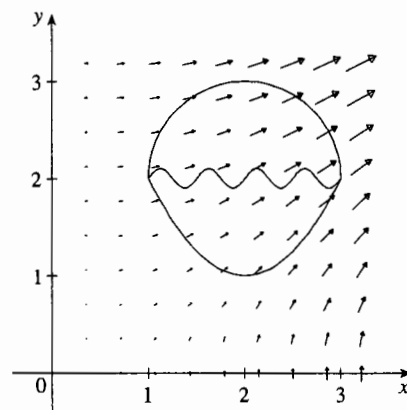
3. $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$
4. $\mathbf{F}(x, y) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$
5. $\mathbf{F}(x, y) = e^x \cos y \mathbf{i} + e^x \sin y \mathbf{j}$
6. $\mathbf{F}(x, y) = (3x^2 - 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$
7. $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$
8. $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (\ln y + 2xy^3)\mathbf{i} + (3x^2y^2 + x/y)\mathbf{j}$

10. $\mathbf{F}(x, y) = (xy \cosh xy + \sinh xy)\mathbf{i} + (x^2 \cosh xy)\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.

- (a) Explain why $\int_C \mathbf{F} \cdot dr$ has the same value for all three curves.
- (b) What is this common value?



- 12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot dr$ along the given curve C .

12. $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$,
 C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$
13. $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$,
 $C: \mathbf{r}(t) = \langle t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t \rangle, \quad 0 \leq t \leq 1$
14. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$,
 $C: \mathbf{r}(t) = \cos t\mathbf{i} + 2 \sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2$
15. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$,
 C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

16. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$,
 $C: x = \sqrt{t}, y = t + 1, z = t^2, 0 \leq t \leq 1$

17. $\mathbf{F}(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$,
 $C: \mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (t^2 - 2t)\mathbf{k}, 0 \leq t \leq 2$

18. $\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\mathbf{k}$,
 $C: \mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq \pi/2$

19–20 Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy$,
 C is any path from $(1, 0)$ to $(2, 1)$

20. $\int_C \sin y dx + (x \cos y - \sin y) dy$,
 C is any path from $(2, 0)$ to $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field \mathbf{F} to move a particle from one point to another point. You decide to check first whether \mathbf{F} is conservative, and indeed it turns out that it is. How would you reply to the request?

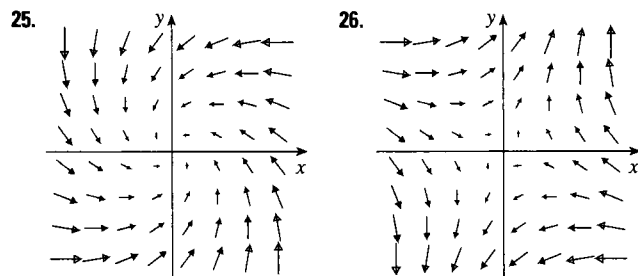
22. Suppose an experiment determines that the amount of work required for a force field \mathbf{F} to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by \mathbf{F} in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about \mathbf{F} ? Why?

23–24 Find the work done by the force field \mathbf{F} in moving an object from P to Q .

23. $\mathbf{F}(x, y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}; P(1, 1), Q(2, 4)$

24. $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}; P(0, 1), Q(2, 0)$

25–26 Is the vector field shown in the figure conservative? Explain.



CAS 27. If $\mathbf{F}(x, y) = \sin y\mathbf{i} + (1 + x \cos y)\mathbf{j}$, use a plot to guess whether \mathbf{F} is conservative. Then determine whether your guess is correct.

28. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral $\int_C y dx - x dy + xyz dz$ is not independent of path.

31–34 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31. $\{(x, y) \mid 0 < y < 3\}$ 32. $\{(x, y) \mid 1 < |x| < 2\}$

33. $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$

34. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

[Hint: Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

36. (a) Suppose that \mathbf{F} is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c , where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the work done by \mathbf{F} in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 4 in Section 16.1. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m²/kg².)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = eqQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 5 in Section 16.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\epsilon = 8.985 \times 10^9$.)