Math 3298 Exam 1
NAME: **SCORE:**

1. Given points $O(0,0,0), A(1,1,1),\ B(1,2,3),\ C(2,-1,2)$.

   (a) (2 pt) Find vectors $\vec{OA}, \vec{OB}, \vec{OC}$.

   $\vec{OA} = \langle 1, 1, 1 \rangle \quad \vec{OB} = \langle 1, 2, 3 \rangle \quad \vec{OC} = \langle 2, -1, 2 \rangle$

   (b) (6 pt) Find length of $\vec{AB}$ and the volume of the parallelepiped spanned by $\vec{OA},\ \vec{OB},\ \vec{OC}$.

   $\vec{AB} = \langle 0, 1, 2 \rangle \quad \boxed{|\vec{AB}| = \sqrt{5}}$

   $\vec{OA} \times \vec{OB} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1, -2, 1 \rangle$

   $V = (\vec{OA} \times \vec{OB}) \cdot \vec{OC} = \langle 1, -2, 1 \rangle \cdot \langle 2, -1, 2 \rangle = \boxed{6}$

2. (7 pt) Find the equation of a line passing point P(4,-1,2) and Q(1,1,5).

   dir : $\vec{PQ} = \langle -3, 2, 3 \rangle$ pt $(4, -1, 2)$

   $$\begin{cases} x = 4 - 3t \\ y = -1 + 2t \\ z = 2 + 3t \end{cases}$$

3. (7 pt) Find equation of the plane which pass through point $P(2,1,1)$ and parallel to another plane $x + 2y + 3z = 0$.

   normal dir $\langle 1, 2, 3 \rangle$ pt $(2, 1, 1)$

   $\boxed{(x-2) + 2(y-1) + 3(z-1) = 0}$

4. (7 pt) Find the equation of a line perpendicular to the plane $3(x-1) + 5(y+2) - (z-3) = 0$ and passing through the point P(1,0,2).

   line parallel to the normal dir of the plane

   dir : $\langle 3, 5, -1 \rangle$. pt $(1, 0, 2)$

   $$\begin{cases} x = 1 + 3t \\ y = 5t \\ z = 2 - t \end{cases}$$

5. (7 pt) Find the point in which the line $x = 1 - t,\ y = 1 + 3t,\ z = t$ intersects the plane $2x - y + z = 2$.

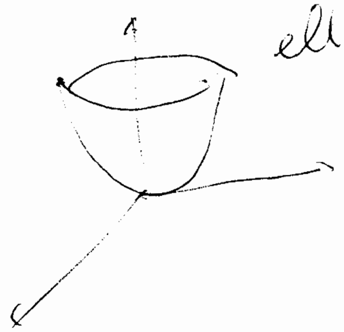
The insection pt is on both line and plane.

$$\begin{cases} x = 1-t,\ y = 1+3t,\ z = t \\ 2x - y + z = 2 \end{cases} \Rightarrow 2(1-t) - (1+3t) + t = 2$$

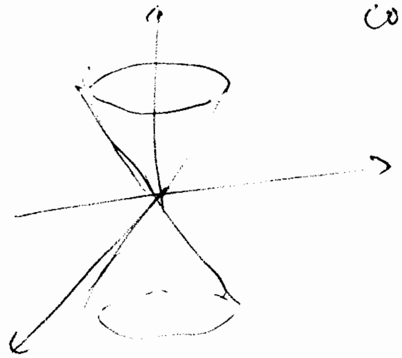
$$\Rightarrow -4t + 1 = 2 \Rightarrow t = -\frac{1}{4}$$

$\boxed{\text{pt } \left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right)}$

6. (7 pt) Make a rough sketch of the quadratic surfaces, and give their names (i) $z = 4x^2 + 9y^2$. (ii) $x^2 + y^2 - z^2 = 0$.



elliptic paraboloid



cone

7. (4 pt) If $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$, then one of them must be a zero vector. True or False. If true, prove it. If false, give an example.

True. If $\vec{a}, \vec{b} \neq \vec{0}$ then $\begin{cases} \vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} \parallel \vec{b} \\ \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b} \end{cases}$

This is impossible

Thus $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. ~~[illegible]~~

8. (7 pt) Find the curve length of the ellipse $\vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle$ for $t \in [0, \pi/3]$.

$$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$$

$$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$$

$$\text{curve length} = \int_0^{\frac{\pi}{3}} |\vec{r}'(t)|\, dt = 3 \cdot \frac{\pi}{3} = \pi$$

9. (7 pt) Find curvature of the curve $y = x^4$ at point $(1, 1)$.

$$K = \frac{f''(x)}{(1+[f'(x)]^2)^{3/2}} = \frac{12x^2}{(1+(4x^3)^2)^{3/2}}$$

at $x=1$
$$\boxed{K = \frac{12}{(17)^{3/2}}}$$

10. (7 pt) A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Find acceleration $\mathbf{a}(t)$, and the tangential and normal components of the acceleration a_T, a_N.

$$\vec{v}(t) = \vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle \qquad v = |\vec{v}(t)| = \sqrt{2}$$
$$\vec{a}(t) = \vec{v}'(t) = \langle -\cos t, -\sin t, 0\rangle$$

$$\boxed{a_T = v' = 0}$$

$$\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)| = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$$
$$\vec{T}'(t) = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle \Rightarrow |\vec{T}'(t)| = \frac{1}{\sqrt{2}} \Rightarrow K = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

$$\boxed{a_N = Kv^2 = \frac{1}{2}\cdot(\sqrt{2})^2 = 1}$$

11. (7 pt) In above problem, find $\mathbf{T}(t), \mathbf{N}(t)$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$$
$$\vec{T}'(t) = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle$$
$$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0\rangle}$$

12. (7 pt) A particle starts at the origin with initial velocity $\mathbf{v}(0) = \langle 0, 0, 0\rangle$. Its acceleration is $\mathbf{a}(t) = \langle 2, t, e^t\rangle$. Find the position vector $\mathbf{r}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t, \tfrac{1}{2}t^2, e^t\rangle + \langle c_1, c_2, c_3\rangle$$

Since $\vec{v}(0) = \vec{0} \Rightarrow \langle 0, 0, 0\rangle = \langle 0, 0, 1\rangle + \langle c_1, c_2, c_3\rangle$

$$\Rightarrow \quad \vec{v}(t) = \langle 2t, \tfrac{1}{2}t^2, e^t - 1\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2, \tfrac{1}{6}t^3, e^t - t\rangle + \langle c_1, c_2, c_3\rangle$$

since $\vec{r}(0) = \vec{0} \Rightarrow \langle c_1, c_2, c_3\rangle = \langle 0, 0, 1\rangle$

$$\boxed{\vec{r}(t) = \langle t^2, \tfrac{1}{6}t^3, e^t - t + 1\rangle}$$

13. (9 pt) Find the local max and min values and saddle points of the function $f(x, y) = x^2 + xy + y^2 + x$.

$$\begin{cases} f_x = 2x + y + 1 = 0 \\ f_y = x + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{2}{3} \\ y = \frac{1}{3} \end{cases} \quad \text{c.p. } \left(-\frac{2}{3}, \frac{1}{3}\right)$$

$f_{xx} = 2 > 0$, $f_{yy} = 2$ $\quad f_{xy} = 1$

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$

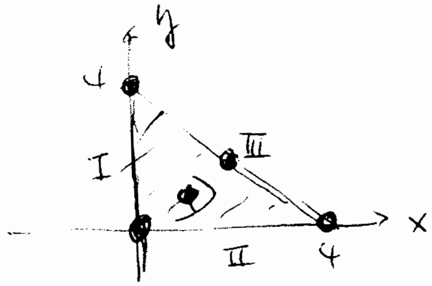
Thus, by 2nd derivative test

$$f\left(-\frac{2}{3}, \frac{1}{3}\right) = \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)$$

$$= \frac{4-2+1}{9} - \frac{6}{9} = -\frac{1}{3}$$

is a local min

14. (9 pt) Find absolute max and min of $f(x, y) = x + y - xy$ on the closed region D bounded by the triangle with vertices $(0, 0), (0, 4), (4, 0)$



set
$$\begin{cases} \frac{\partial f}{\partial x} = 1 - y = 0 \\ \frac{\partial f}{\partial y} = 1 - x = 0 \end{cases} \Rightarrow (1, 1) \text{ is c.p.}, \quad (1, 1) \in D$$

on boundary I, $x = 0$, $0 \le y \le 4$

$f(0, y) = y$, possible extreme pt at $y = 0$, $y = 4 \Rightarrow (0, 0), (0, 4)$

on boundary II, $y = 0$, $0 \le x \le 4$

$f(x, 0) = x$, possible extreme pt at $x = 0$, $x = 4 \Rightarrow (0, 0), (4, 0)$

on boundary III, $x + y = 4 \Rightarrow y = 4 - x$

$g(x) = f(x, 4 - x) = x + (4 - x) - x(4 - x) = 4 - 4x + x^2$ set $g'(x) = 2x - 4 = 0 \Rightarrow x = 2$

possible extreme pt at $x = 0, 2, 4 \Rightarrow (0, 4), (2, 2), (4, 0)$

$\boxed{f(0,0) = 0}$ $\boxed{f(0,4) = 4}$ $\boxed{f(4,0) = 4}$ $f(1,1) = 1$ $\boxed{f(2,2) = 0}$

abs min $\quad$ abs max $\quad$ abs max $\quad$ abs min

Mean	77.8
Median	83.00
90–100	11
highest	98