Math 3298 Exam II
NAME: **SCORE:**

9 pt 1. Evaluate the iterated integral

$$\int_0^2 \int_{x^2}^4 xydydx.$$

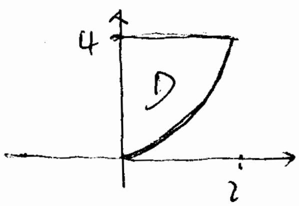
$$= \int_0^2 \frac{x}{2} y^2 \Big|_{x^2}^4 dx$$

$$= \int_0^2 \frac{x}{2}(16 - x^4)\,dx$$

$$= \int_0^2 \left(8x - \frac{1}{2}x^5\right) dx$$

$$= \left(4x^2 - \frac{1}{12}x^6\right)\Big|_0^2 = 16 - \frac{64}{12} = 16 - \frac{16}{3} = \boxed{\frac{32}{3}}$$

9 pt 2. Sketch the region D of the double integral associated with the iterated integral in last problem. Then change of order of the iterated integral to $dxdy$. Evaluate it.

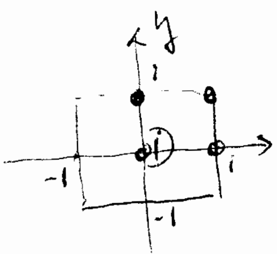


$$\int_0^4 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^4 \frac{y}{2} x^2 \Big|_0^{\sqrt{y}} dy = \int_0^4 \frac{y^2}{2} dy = \frac{1}{6} y^3 \Big|_0^4$$

$$= \frac{64}{6} = \boxed{\frac{32}{3}}$$

9 pt 3. Use a Riemann sum with $m = n = 2$ to estimate the double integreal $\int\int_D \frac{1}{2+x+y^2} dA$ where $D = [-1,1] \times [-1,1]$. The sample points are chosen to be the right upper corner of each sub-region.



$$\Delta x = \Delta y = \frac{1-(-1)}{2} = 1 \qquad \Delta A = \Delta x \Delta y = 1$$

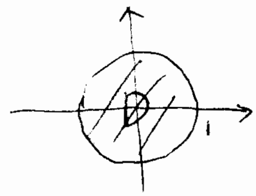
$$\iint_D \frac{1}{2+x+y^2} dA \approx \left(\frac{1}{2+0+0^2} + \frac{1}{2+1+0^2} + \frac{1}{2+0+1^2}\right.$$

$$\left. + \frac{1}{2+1+1^2}\right) \cdot 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{2}{3} + \frac{3}{4} = \boxed{\frac{17}{12}}$$

27

9 pt 4. Find the area of the surface $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

$f(x,y) = xy. \qquad f_x = y, \quad f_y = x$



$$A(S) = \iint_D \sqrt{1 + y^2 + x^2}\, dA$$

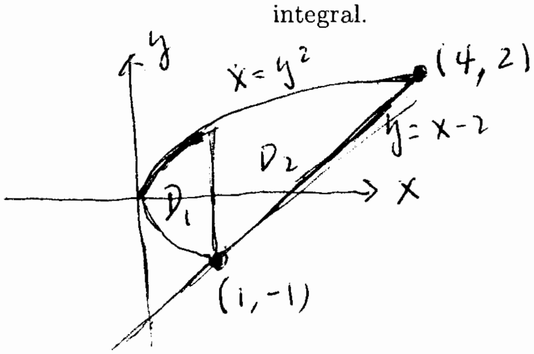
$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\; r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{1}{3}(1+r^2)^{3/2} \Big|_0^1 = \frac{2\pi}{3}\left(2^{3/2} - 1\right) = \boxed{\frac{2\pi}{3}(\sqrt{8} - 1)}$$

9 pt 5. Given

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy dx.$$

Sketch the region of integration. Then change the integration order and evaluate the new integral.



$$\int_{-1}^{2} \int_{y^2}^{y+2} dx\, dy = \int_{-1}^{2} (y + 2 - y^2)\, dy$$

$$= \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_{-1}^{2} = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 5 - \frac{1}{2} = \boxed{\frac{9}{2}}$$

9 pt 6. Find the volume of the region below the surface $z = 1 + e^x \sin y$, above $z = 0$, and surrounded by planes $x = 0, x = 1, y = 0, y = \pi$.

$$V = \iint_D (1 + e^x \sin y)\, dA = \int_0^1 \int_0^{\pi} (1 + e^x \sin y)\, dy\, dx$$

$$= \int_0^1 (y - e^x \cos y)\Big|_0^{\pi} dx = \int_0^1 \left[(\pi + e^x) - (0 - e^x)\right] dx$$

$$= (\pi x + 2e^x)\Big|_0^1 = (\pi + 2e) - (0 + 2) = \boxed{\pi + 2e - 2}$$

/27

8 pt 7. Use Lagrange multiplier method to find the minimum/maximum of the product of three nonnegative real numbers whose sum equals 100.

$f(x,y,z) = xyz$ $\quad \nabla f = \langle yz, xz, xy \rangle$ $\quad \nabla g = \langle 1, 1, 1 \rangle$

Constraint $\quad x+y+z=100$

$$\begin{cases} yz = \lambda & (1) \\ xz = \lambda & (2) \\ xy = \lambda & (3) \\ x+y+z=100 & (4) \end{cases}$$

From (1),(2) $\quad yz = xz \Rightarrow (y-x)z = 0 \quad (5)$

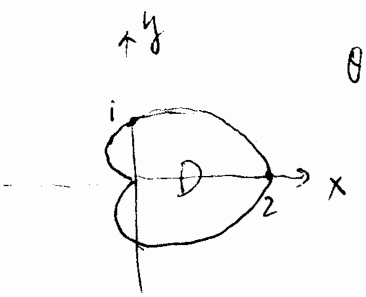
From (2),(3) $\quad xz = xy \Rightarrow (z-y)x = 0 \quad (6)$

when $z \neq 0$, from (5) $y = x$

when $x \neq 0$ from (6) $z = y$) plug into (4)

$x = y = z = \frac{100}{3} \Rightarrow \max = \left(\frac{100}{3}\right)^3$ $\quad$ min $= 0$ when one of $x, y, z = 0$.

9 pt 8. Use double integral in polar coordinates to find the area inside the cardioid $r = 1 + \cos\theta$. (Sketch the region first to determine the limit of integration)



$\theta \in [0, 2\pi]$

$$A(D) = \iint_D 1 \cdot dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{1}{2}\cdot\frac{3}{2}\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{3\pi}{2}}$$

9 pt 9. Find the volume of the solid that lies within the cylinder $x^2 + y^2 = 1$, and between the plane $z = 0$ and the top half sphere $z = \sqrt{4 - x^2 - y^2}$.

$$V = \iiint_R 1 \cdot dV = \iint_D \sqrt{4-x^2-y^2}\,dA \qquad D = \{(x,y) \mid x^2+y^2 \le 1\}$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4-r^2}\,r\,dr\,d\theta$$

$$= 2\pi\,(4-r^2)^{3/2}\left(-\frac{1}{3}\right)\Big|_0^1 = -\frac{2\pi}{3}\left(3^{3/2} - 4^{3/2}\right)$$

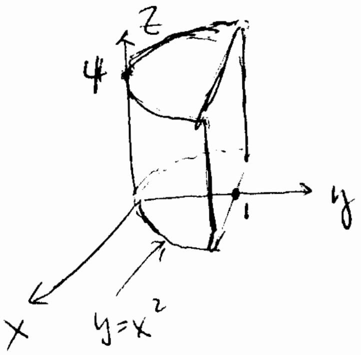
$$= \frac{2\pi}{3}\left(8 - \sqrt{27}\right)$$

26

5 pt 10. Given the following iterated integrals

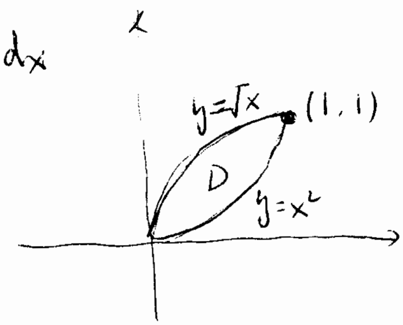
$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{4} f(x,y,z)\,dz\,dy\,dx.$$

Sketch the region of integration. Then change the integration order to $dydzdx$.



$$\int_{-1}^{1}\int_{0}^{4}\int_{x^2}^{1} f(x,y,z)\,dy\,dz\,dx$$

15 pts 11. Find the mass, moments about x, y axis, and center of mass for a lamina that occupies the region D which is bounded by $y = x^2$ and $x = y^2$. The density is $\rho(x,y) = \sqrt{x}$.



$$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} \sqrt{x}\,dy\,dx = \int_0^1 \sqrt{x}(\sqrt{x}-x^2)\,dx$$

$$= \frac{1}{2}x^2 - \frac{2}{7}x^{\frac{7}{2}}\Big|_0^1 = \frac{1}{2}-\frac{2}{7} = \boxed{\frac{3}{14}}$$

$$M_x = \iint_D y\rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} y\sqrt{x}\,dy\,dx$$

$$= \int_0^1 \sqrt{x}\cdot\frac{1}{2}y^2\Big|_{x^2}^{\sqrt{x}}dx = \frac{1}{2}\int_0^1 \sqrt{x}\,(x - x^4)\,dx = \frac{1}{2}\int_0^1 (x^{3/2} - x^{\frac{9}{2}})\,dx$$

$$= \frac{1}{2}\left(\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{11}x^{\frac{11}{2}}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{2}{5}-\frac{2}{11}\right) = \boxed{\frac{6}{55}}$$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} x\sqrt{x}\,dy\,dx = \int_0^1 x^{3/2}(x^{\frac{1}{2}} - x^2)\,dx$$

$$= \int_0^1 (x^2 - x^{\frac{7}{2}})\,dx = \frac{1}{3}x^3 - \frac{2}{9}x^{9/2}\Big|_0^1 = \frac{1}{3}-\frac{2}{9} = \boxed{\frac{1}{9}}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{9}}{\frac{3}{14}} = \frac{14}{27} \qquad \bar{y} = \frac{M_x}{m} = \frac{\frac{6}{55}}{\frac{3}{14}} = \boxed{\frac{28}{55}}$$

20