Math 3298 Exam III
NAME: **SCORE:**

1. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $C : \mathbf{r}(t) = \langle 2t, 3t, -t^2\rangle, -1 \le t \le 1$, and $\mathbf{F} = \langle x, -z, y\rangle$.

$$\int_C F\cdot dr = \int_C x\,dx - z\,dy + y\,dz = \int_{-1}^{1} (2t)2\,dt + (t^2)3\,dt + (3t)(-2t)\,dt$$

$$= \int_{-1}^{1} (4t - 3t^2)\,dt = 2t^2 - t^3 \Big|_{-1}^{1} = (-1)+(-1)^3 = -2$$

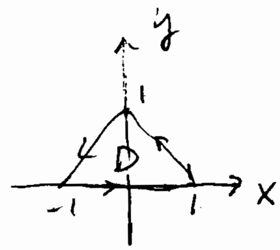
2. Use Green's Theorem to evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (e^{x^2} + y)dx + (2x + \cos y)dy$$

where C is the triangle with vertices $(-1, 0), (1, 0)$, and $(0, 1)$ with positive orientation.

$$\frac{\partial Q}{\partial x} = 2 \qquad \frac{\partial P}{\partial y} = 1$$

$$\int_C F\cdot dr = \iint_D 1\cdot dA = A(D) = \frac{1}{2}\cdot 2\cdot 1 = 1$$



3. Given vector field $\mathbf{F} = \langle 2xy^3, 3x^2y^2\rangle$. Show it is conservative. Then evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where C is a curve starts at $(1, 1)$ and ends at $(2, 3)$.

$$\frac{\partial Q}{\partial x} = 6xy^2 \qquad \frac{\partial P}{\partial y} = 6xy^2 \Rightarrow \text{conservative}$$

$$\int_C \vec{F}\cdot d\vec{r} = f(2,3) - f(1,1) = 2^2\cdot 3^3 - 1^2\cdot 1^3 = \boxed{107}$$

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy^3 \\ \frac{\partial f}{\partial y} = 3x^2y^2 \end{cases} \Rightarrow f(x,y) = \int 2xy^3\,dx = x^2y^3 + g(y) \Rightarrow \frac{\partial f}{\partial y} = 3x^2y^2 + g'(y)$$

Thus, $g'(y) = 0 \Rightarrow g(y) = 0$

$$\Rightarrow \boxed{f(x,y) = x^2y^3}$$

4. Find a parametric equation for the curve $y = \sin x$ starting from $(0,0)$ and ending at $(\pi, 0)$. Specify the parameter region.

$$\begin{cases} x = t \\ y = \sin t \end{cases} \qquad t \in [0, \pi]$$

5. Given a parametric surface

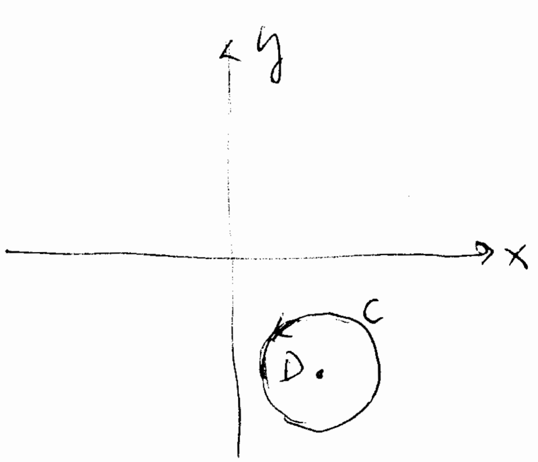
$$\mathbf{r}(u,v) = \langle 1+u, 2+v, 5+u-v \rangle.$$

Find its Cartesian equation. What kind of surface is it?

$$\begin{cases} x = 1+u \\ y = 2+v \\ z = 5+u-v \end{cases} \Rightarrow z = 5 + (x-1) - (y-2)$$

$\Rightarrow$ plane.

6. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle y - \cos y, x \sin y \rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$.



$$\frac{\partial Q}{\partial x} = \sin y. \qquad \frac{\partial P}{\partial y} = 1 + \sin y$$

By Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= -\iint_D dA = -\pi \cdot 2^2$$

$$= -4\pi$$

7. Compute the line integral

$$\int_C yz\cos x\,ds, \qquad C: x=t,\ y=3\cos t,\ z=3\sin t,\ 0\le t\le \pi.$$

$$= \int_0^{\pi} (3\cos^2 t)(3\sin t)\sqrt{1^2+(-3\sin t)^2+(3\cos t)^2}\,dt$$

$$= 9\sqrt{10}\int_0^{\pi} \cos^2 t \sin t\,dt = 9\sqrt{10}\left(-\frac{1}{3}\right)\cos^3 t\Big|_0^{\pi}$$

$$= -3\sqrt{10}\left(\cos^3\pi - \cos^3 0\right) = -3\sqrt{10}(-1-1) = \boxed{6\sqrt{10}}$$

8. Evaluate the triple integral using spherical coordinates

$$\iiint_E (x^2+y^2)\,dV$$

where E is the region between two spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$

$$\iiint_E (x^2+y^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 \left(\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin^3\phi \cdot \frac{1}{5}\rho^5\Big|_2^3\,d\theta\,d\phi = \frac{1}{5}(3^5-2^5)\cdot 2\pi\int_0^{\pi}\sin^2\phi\sin\phi\,d\phi$$

$$= \frac{1}{5}(243-32)\cdot 2\pi\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$= \frac{422}{5}\pi\cdot\left(-\cos\phi+\frac{1}{3}\cos^3\phi\right)\Big|_0^{\pi} = \frac{422}{5}\pi\left[\left(+1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= \left(\frac{422}{5}\pi\right)\cdot\frac{4}{3}$$

9. Given a vector field $\mathbf{F} = \langle xy, yz, zx\rangle$. Define curl$\mathbf{F}$ and div$\mathbf{F}$.

$$\text{curl}\,F = \langle -y, -z, -x\rangle$$

$$\text{div}\,F = y+z+x.$$

10. Check whether the vector field

$$\mathbf{F} = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle$$

is conservative. If so, find the potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

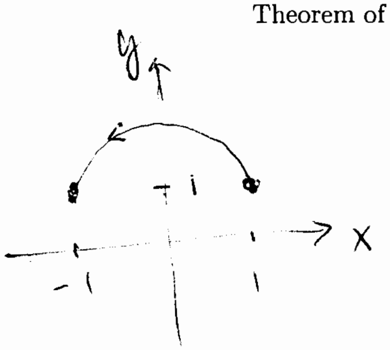
$\frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}$, $\frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2 \Rightarrow \vec{F}$ is conservative

$$\begin{cases} f_x = \ln y + 2xy^3 \\ f_y = 3x^2y^2 + \frac{x}{y} \end{cases} \Rightarrow f(x,y) = x \ln y + x^2y^3 + g(y) \Rightarrow f_y = \frac{x}{y} + 3x^2y^2 + g'(y)$$

Thus. $g'(y) = 0 \Rightarrow g(y) = C$

$$f(x,y) = x \ln y + x^2y^3 + C$$

11. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the $\mathbf{F}$ given in last problem by the Fundamental Theorem of Calculus , where C is the top half circle $x^2 + (y-1)^2 = 1$ in positive orientation.



$$\int_C \vec{F} \cdot d\vec{r} = f(-1,1) - f(1,1)$$
$$= \left[(-1)\ln 1 + (-1)^2 \cdot 1^3\right] - \left[1 \cdot \ln 1 + 1^2 \cdot 1^3\right]$$
$$= 0$$

Evaluate the last problem by the Green's Theorem. Hint: add a curve C_1 which is the horizontal line segment from (-1, 1) to (1,1) so that $C \cup C_1$ enclose a region D.

Since $0 = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA = \int_{C \cup C_1} \vec{F} \cdot d\vec{r}$,

We have $\int_C \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{C_1} P\,dx + Q\,dy$.

$$= -\int_{-1}^{1} (\ln 1 + 2t \cdot 1^3)\,dt = t^2 \Big|_{-1}^{1} = 0.$$

$C_1 : \begin{cases} x = t \\ y = 1 \end{cases}$

$-1 \le t \le 1$

$\Rightarrow dy = 0$