Math 3298 Exam 1
NAME: **SCORE:**

1. Given three points $A(1,1,1)$, $B(1,2,3)$, $C(2,-1,2)$.

   (a) Find vectors $\vec{AB}$, $\vec{AC}, \vec{BC}$.

   $\vec{AB} = \langle 0,1,2 \rangle$, $\vec{AC} = \langle 1,-2,1 \rangle$ $\vec{BC} = \langle 1,-3,-1 \rangle$

   (b) Find $\vec{AB} + \vec{BC}$, $\vec{AB} - \vec{AC}$, and $2\vec{AB}$.

   $\vec{AB} + \vec{BC} = \langle 1,-2,1 \rangle$

   $\vec{AB} - \vec{AC} = \langle -1,3,1 \rangle$

   $2\vec{AB} = \langle 0,2,4 \rangle$

   (c) Find length of $\vec{AB}$ and area of the parallelogram spanned by $\vec{AB}$, $\vec{AC}$.

   $|\vec{AB}| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$

   $\text{area} = |\vec{AB} \times \vec{AC}| = \left|\left\langle \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} \right\rangle\right|$

   $= |\langle 5, 2, -1 \rangle| = \sqrt{5^2+2^2+(-1)^2} = \sqrt{30}$

2. Given two unit vectors $\mathbf{u}, \mathbf{v}$. The angle between them is $\pi/4$. What is $\mathbf{u} \cdot \mathbf{v}$?

   $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\frac{\pi}{4} = 1 \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

3. Find the equation of a line passing point $P(1,1,5)$ and parallel to another line whose equation is $x = 1+t,\ y = 3-5t,\ z = 9+2t$.

   dir $\vec{v} = \langle 1,-5,2 \rangle$

   line eqn, $\vec{r}(t) = \langle 1,1,5 \rangle + t\langle 1,-5,2 \rangle$ $\forall t$

   or $\begin{cases} x = 1+t \\ y = 1-5t \\ z = 5+2t \end{cases}$ $\forall t$.

4. Find equation of the plane which pass through point $P(2,1,1)$ and parallel to another plane $x + 2y - z = 0$.

normal dir $\vec{n} = \langle 1, 2, -1 \rangle$

plane eqn: $\boxed{1\cdot(x-2) + 2(y-1) - 1\cdot(z-1) = 0}$

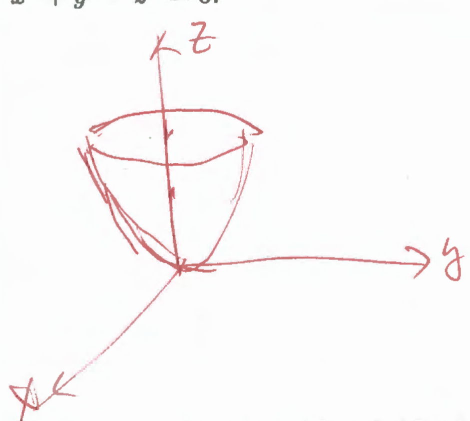
5. Find the point in which the line $x = 1-t,\ y = 1+3t,\ z = t$ intersects the plane $2x-y+z=2$.

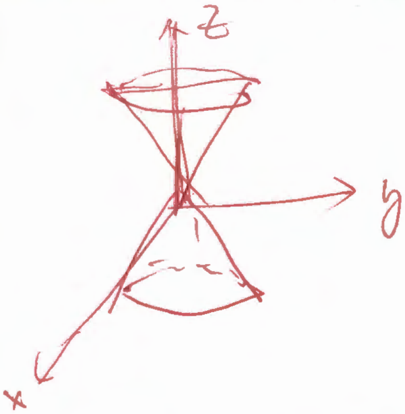
At the intersection: $2(1-t)-(1+3t)+t = 2$

$-4t = 2-2+1 \Rightarrow t = -\frac{1}{4}$

Thus $\begin{cases} x = \frac{5}{4} \\ y = \frac{1}{4} \\ z = -\frac{1}{4} \end{cases}$ point is $\left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right)$

6. Make a rough sketch of the quadratic surfaces, and give their names (i) $z = 4x^2 + 9y^2$. (ii) $x^2 + y^2 - z^2 = 0$.





7. Find a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for the curve which is the intersection of surfaces $4 = x^2 + y^2$ and $z^2 = x^2 + y^2$.

$\langle x(t), y(t), z(t) \rangle = \langle 2\cos t, 2\sin t, \pm 2 \rangle$

$0 \le t \le 2\pi$

8. Suppose $\mathbf{a} \neq \mathbf{0}$. If $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, can we claim that $\mathbf{b} = \mathbf{c}$? If Yes, prove it. If no, give an example to support your conclusion.

No! If $(\vec{b}-\vec{c}) \parallel \vec{a}$, then $\vec{a} \times (\vec{b}-\vec{c}) = \vec{0}$

ex. $\vec{a} = \langle 1, 0, 0 \rangle$. $\vec{b} = \langle 1, 1, 1 \rangle$. $\vec{c} = \langle 0, 1, 1 \rangle$

$\vec{a} \times (\vec{b}-\vec{c}) = \langle 1, 0, 0 \rangle \times \langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle$

But $\vec{b} \neq \vec{c}$.

9. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Find its velocity, speed, and acceleration.

$\cancel{v = |\vec{v}(t)| = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}} = \langle \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$

$\cancel{v = |\vec{v}(t)| =}$ $\boxed{\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle}$

$\boxed{\vec{a}(t) = \vec{r}\,''(t) = \langle -\cos t, -\sin t, 0 \rangle}$

$\boxed{v = |\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}}$

10. In above problem, find the tangential and normal components of the acceleration. Also, find $\mathbf{T}(t), \mathbf{N}(t)$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \left\langle \begin{vmatrix} \cos t & 1 \\ -\sin t & 0 \end{vmatrix}, -\begin{vmatrix} -\sin t & 1 \\ -\cos t & 0 \end{vmatrix}, \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \right\rangle$

$= \langle \sin t, -\cos t, 1 \rangle$.

$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{2} \Rightarrow k = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$

$\boxed{a_T = v' = 0, \quad a_N = kv^2 = \frac{1}{2} \cdot 2 = 1}$ $\boxed{\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$

$\vec{N} = \vec{T}\,'/|\vec{T}\,'| = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \rangle / \frac{1}{\sqrt{2}} = \boxed{\langle -\cos t, -\sin t, 0 \rangle = \vec{N}}$

11. A particle starts at the origin with initial velocity $\mathbf{v}(0) = \langle 0, 0, 0 \rangle$. Its acceleration is $\mathbf{a}(t) = \langle 2, 1, 0 \rangle$. Find the distance it travels from $t = 0$ to $t = 1$.

$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 2, 1, 0 \rangle\,dt + \vec{c} = \langle 2t, t, 0 \rangle + \vec{c}$

since $\vec{v}(0) = \langle 0, 0, 0 \rangle \Rightarrow \vec{c} = \langle 0, 0, 0 \rangle \Rightarrow \vec{v}(t) = \langle 2t, t, 0 \rangle$

$v(t) = \sqrt{(2t)^2 + t^2 + 0^2} = \sqrt{5}\,t$

$s = \int_0^1 \sqrt{5}\,t\,dt = \frac{\sqrt{5}}{2} t^2 \Big|_0^1 = \frac{\sqrt{5}}{2}$

12. Find the local max and min values and saddle points of the function $f(x,y) = x^2 + xy + y^2 + x$.

$$\begin{cases} f_x = 2x + y + 1 = 0 \\ f_y = x + 2y = 0 \end{cases} \Rightarrow \text{only c.p. } \left(-\frac{2}{3}, \frac{1}{3}\right)$$

$f_{xx} = 2$, $f_{yy} = 2$. $f_{xy} = f_{yx} = 1$

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$

$\boxed{\text{Thus } f\left(-\frac{2}{3}, \frac{1}{3}\right) = -\frac{1}{3} \text{ is a local min, no local max, no saddle pt}}$

13. (i) Find ∇f for $f(x,y,z) = \sin x \cos y e^z$. (ii) Find the directional derivative of f along $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$. (3) At point $(\pi, \pi, 0)$, find the direction which will yield the largest directional derivative of f

$$\nabla f = \langle \cos x \cos y\, e^z, -\sin x \sin y\, e^z, \sin x \cos y\, e^z \rangle$$

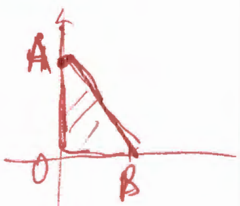
$$D_u f = \nabla f \cdot \vec{u} = \frac{1}{\sqrt{2}} \cos x \cos y \cdot e^z + \frac{1}{\sqrt{2}} \sin x \cos y\, e^z$$

$$= \frac{1}{\sqrt{2}} \cos y \cdot e^z (\cos x + \sin x)$$

$\boxed{\nabla f \text{ at } (\pi, \pi, 0)} = \langle (-1)(-1)e^0, 0, 0 \rangle = \langle 1, 0, 0 \rangle$

$D_{\vec{u}} f$ reach max when $\vec{u} = \frac{\nabla f}{|\nabla f|}(\pi, \pi, 0) = \langle 1, 0, 0 \rangle$

14. Find absolute max and min of $f(x,y) = x + y - xy$ on the closed region D bounded by the triangle with vertices $(0,0), (0,2), (4,0)$



$$\begin{cases} f_x = 1 - y = 0 \\ f_y = 1 - x = 0 \end{cases} \quad \text{c.p } (1,1) \Rightarrow \boxed{f(1,1) = 1 + 1 - 1 = 1}$$

on OB, $f(x, 0) = x$, $\boxed{f(0,0) = 0. \quad f(4,0) = 4}$

on OA $f(0, y) = y$ $\boxed{f(0,0) = 0 \quad f(0,2) = 2}$

on AB $\underbrace{f\left(x, -\frac{1}{2}x + 2\right)}_{g(x)} = x + \left(-\frac{1}{2}x + 2\right) - x\left(-\frac{1}{2}x + 2\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2$

$y = -\frac{1}{2}x + 2$

c.p of $g(x)$ in $(0, 4)$ is

$g'(x) = 0 \Rightarrow x - \frac{3}{2} = 0 \Rightarrow y = \frac{5}{4}, \; x = \frac{3}{2}$

$\boxed{f\left(\frac{3}{2}, \frac{5}{4}\right) = 7/8}$

Math 3298 Exam 2
NAME: SCORE:

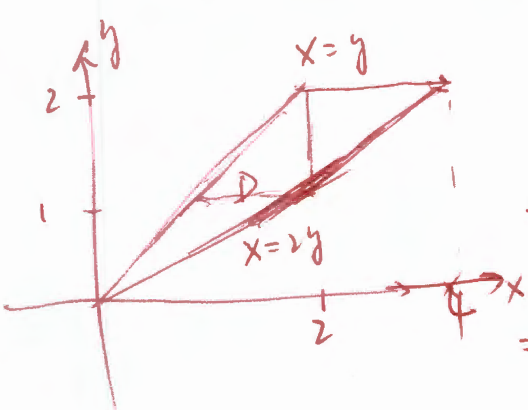
1. Evaluate the iterated integral

$$\int_0^2\int_y^{2y} x^2y\,dx\,dy.$$

$$= \int_0^2 y\cdot\frac{x^3}{3}\Big|_y^{2y}dy = \frac13\int_0^2 y(8y^3-y^3)\,dy$$

$$= \frac73\int_0^2 y^4\,dy = \frac73\cdot\frac{32}{5} = \frac{224}{15}$$

2. Sketch the region D of the double integral associated with the iterated integral in last problem. Then change of order of the iterated integral to $dydx$. Evaluate it.



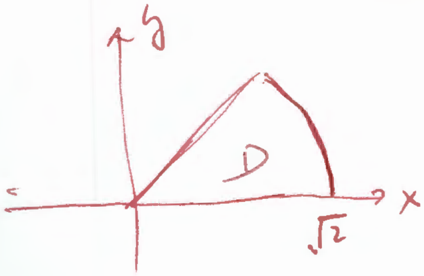
$$\int_0^2\int_{x/2}^{x} x^2y\,dy\,dx + \int_2^4\int_x^2 x^2y\,dy\,dx$$

$$= \int_0^2 x^2\cdot\frac12 y^2\Big|_{\frac{x}{2}}^{x}dx + \int_2^4 x^2\frac12 y^2\Big|_x^2 dx$$

$$= \int_0^2 \frac12 x^2\cdot\frac34 x^2\,dx + \int_2^4 \frac12 x^2(4-x^2)\,dx$$

$$= \frac38\cdot\frac{32}{5} + \left(\frac23 x^3 - \frac1{10}x^5\right)\Big|_2^4 = \frac{224}{15}$$

3. Find the area of the region D bounded by the curves $x=y$, $x=\sqrt{2-y^2}$, and $y=0$. Sketch the region of integration FIRST.



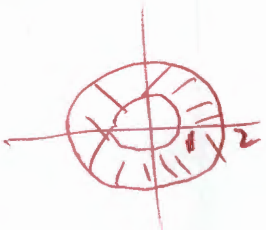
$$\iint_D 1\cdot dA = \int_0^{\frac{\pi}{4}}\int_0^{\sqrt2} r\,dr\,d\theta$$

$$= \frac{\pi}{4}\cdot\frac12 r^2\Big|_0^{\sqrt2} = \frac{\pi}{4}$$

4. A plate lies on the region D bounded between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The density is $\rho(x, y) = \frac{2}{\sqrt{x^2+y^2}}$. Find the mass of the plate. Hint: switch to polar coordinates.

$$m = \iint_D \rho(x,y)\,dA = \int_0^{2\pi}\int_1^2 \frac{2}{\sqrt{r^2}}\, r\,dr\,d\theta$$

$$= 2\pi \cdot 2r\Big|_1^2 = 4\pi$$



5. Find the volume of the solid under the surface $z = 9 - x^2 - y^2$, and over the rectangle $D = [0, 2] \times [0, 2]$.

$$\iint_D (9-x^2-y^2)\,dA = \int_0^2\int_0^2 (9-x^2-y^2)\,dy\,dx$$

$$= \int_0^2 \left(9y - x^2y - \tfrac{1}{3}y^3\right)\Big|_0^2 dx$$

$$= \int_0^2 \left[2(9-x^2) - \tfrac{8}{3}\right] dx = \tfrac{46}{3}x - \tfrac{2}{3}x^3\Big|_0^2$$

$$= \frac{26}{3}$$

6. Find the surface area of the paraboloid $z = x^2 + y^2$ between $z = 1$ and $z = 4$.

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

$$S = \iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\, r\,dr\,d\theta$$

same D as #4

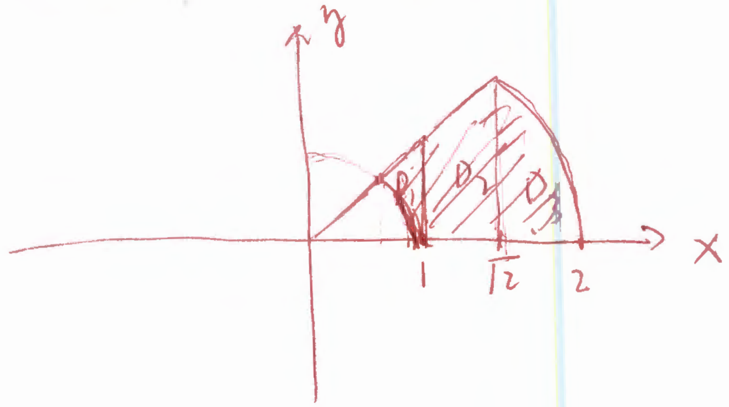
$$= 2\pi \cdot \tfrac{1}{12}(1+4r^2)^{\frac{3}{2}}\Big|_1^2 = \tfrac{\pi}{6}\left((1+16)^{3/2} - (1+4)^{3/2}\right)$$

$$= \tfrac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

7. Given the following double integrals

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx.$$

Sketch all three regions of integration on one XY plane.



8. Use an easy way to calculate the integral in last problem.

$D = D_1 \cup D_2 \cup D_3$

$$\iint_D xy\,dA = \int_0^{\frac{\pi}{4}}\int_1^2 r\cos\theta\, r\sin\theta \cdot r\,dr\,d\theta$$

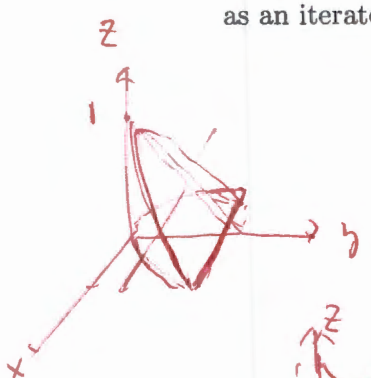
$$= \int_0^{\frac{\pi}{4}} \cos\theta\sin\theta \cdot \frac{1}{4}r^4\Big|_1^2\,d\theta$$

$$= \frac{1}{4}(16-1)\cdot\frac{1}{2}\sin^2\theta\Big|_0^{\frac{\pi}{4}} = \frac{15}{4}\cdot\frac{1}{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{15}{16}$$

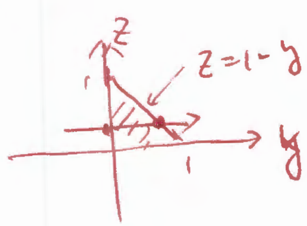
9. Rewrite the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$$

as an iterated integral in the order $dxdydz$. Sketch the region first.



$$\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$



10. Calculate the triple integral $\iiint_E \frac{1}{x^2+y^2+z^2}\,dV$ by converting it to spherical coordinates. Here E is the region between sphere $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \frac{1}{\rho^2}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi \cdot \rho \Big|_1^2 \, d\phi\, d\theta$$

$$= \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} \, d\theta$$

$$= \cancel{8}\ 2\pi\cdot(1+1) = 4\pi$$

Math 3298 Exam III
NAME: **SCORE:**

1. Compute the line integral

$$\int_C yz\cos x\,ds, \qquad C: x=t,\ y=3\cos t,\ z=3\sin t,\ 0\le t\le \pi.$$

$$= \int_0^{\pi} 3\cos t\cdot 3\sin t\cdot \cos t\,\sqrt{1+(-3\sin t)^2+(3\cos t)^2}\,dt$$

$$= 9\sqrt{10}\cdot\int_0^{\pi}\sin t\cos^2 t\,dt = 9\sqrt{10}\left(\frac{-1}{3}\right)\cos^3 t\Big|_0^{\pi}$$

$$= -3\sqrt{10}\cdot(-1-1) = 6\sqrt{10}$$

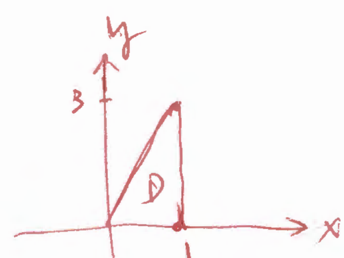
2. Use Green's Theorem to evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ with positive orientation.

$P=\sqrt{1+x^3}$, $\quad \frac{\partial P}{\partial y}=0$

$Q=2xy \qquad \frac{\partial Q}{\partial x}=2y$



By Green's thm $\int_C \vec{F}\cdot d\vec{r} = \iint_D 2y\,dA$

$$= \int_0^1\int_0^{3x} 2y\,dy\,dx = \int_0^1 y^2\Big|_0^{3x}dx$$

$$= \int_0^1 9x^2\,dx = 3x^3\Big|_0^1 = 3$$

3. Find a parametric equation for the curve $y = \sin x$ starting from $(0,0)$ and ending at $(\pi, 0)$. Specify the parameter region.

$$\begin{cases} x = t \\ y = \sin t \end{cases} \qquad 0 \leq t \leq \pi$$

4. Find a parametric equation for the surface which is the part of the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2+y^2}$. Specify the parameter region.

$$\begin{cases} x = 2\cos\theta \sin\varphi \\ y = 2\sin\theta \sin\varphi \\ z = 2\cos\varphi \end{cases} \qquad (\theta, \varphi) \in D = [0, 2\pi] \times [0, \tfrac{\pi}{4}]$$

$$\begin{cases} 2z^2 = 4 \Rightarrow z = \sqrt{2} \Rightarrow \sqrt{2} = 2\cos\varphi \\ \varphi = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \end{cases}$$

5. Given a parametric surface

$$\mathbf{r}(u,v) = \langle u\cos v, u \sin v, \sin u\rangle, \qquad u \in [-\pi, \pi], \quad v \in [0, 2\pi].$$

Find its Cartesian equation.

$$x^2 + y^2 = u^2\cos^2 v + u^2 \sin^2 v = u^2 \Rightarrow u = \pm\sqrt{x^2+y^2}$$

$$z = \sin u = \pm \sin\sqrt{x^2+y^2} = \pm \sin\sqrt{x^2+y^2}$$

6. Given a vector field $\mathbf{F} = \langle p(x,y,z), Q(x,y,z), R(x,y,z)\rangle$. Define curl$\mathbf{F}$ and div$\mathbf{F}$.

$$\text{Curl } \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

7. Suppose that you know that $f(x, y) = e^x + x^2y^2$ and

$$\nabla f = \mathbf{F} = \langle e^x + 2xy^2, 2x^2y\rangle.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C : is $\mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t\rangle, 0 \le t \le 1$.

$$\int_C F \cdot dr = f(r(1)) - f(r(0))$$

$$= f(1, 1) - f(0, 1)$$

$$= (e+1) - 1$$

$$= e$$

8. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$\mathbf{F}(x, y, z) = \langle xz, -2y, 3x\rangle$, $\quad S : \mathbf{r}(u, v) = \langle 2 \sin u \cos v, 2 \sin u \sin v, 2 \cos u\rangle$

and $0 \le u \le \pi$, and $0 \le v \le 2\pi$.

$$\iint_S F \cdot dS = \iint_D \vec{F} \cdot (r_u \times r_v)\, dA = \iint_D \langle 4\sin u\cos u\cos v, -4\sin u\sin v, 6\sin u\cos v\rangle \cdot$$

$$\langle +4\cos v\sin^2 u, +4\sin v\sin^2 u, +4\sin u\cos u\rangle\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(+16\cos^2 v\cos u\sin^3 u + 16\sin^2 v\sin^3 u + 24\cos v\cos u\sin^2 u\right) du\, dv$$

first & third integral $= 0$

$$= \int_0^{2\pi}\int_0^{\pi} 16\sin^2 v\sin^3 u\, du\, dv = 16 \cdot \pi \cdot \frac{8}{3} = \frac{128}{3}\pi$$

9. Check whether the vector field

$$\mathbf{F} = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle$$

is conservative. If so, find the potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

$P = \ln y + 2xy^3$. $\frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2$

$Q = 3x^2y^2 + \frac{x}{y}$ $\frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}$

$\Rightarrow \vec{F}$ conservative

$f(x,y) = \int (\ln y + 2xy^3)\,dx = x\ln y + x^2y^3 + g(y)$

$\frac{\partial f}{\partial y} = \frac{x}{y} + 3x^2y^2 + g'(y) = Q = 3x^2y^2 + \frac{x}{y} \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = 0$.

Thus, $f(x,y) = x\ln y + x^2y^3$

10. Find the surface area of the upper half sphere $x^2 + y^2 + z^2 = 4$ and $z \geq 0$ by surface integral.

S: $r(\theta, \varphi) = \langle 2\cos\theta\sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi \rangle$. $\theta \in [0, 2\pi]$, $\varphi \in [0, \frac{\pi}{2}]$

$A(S) = \iint_S 1\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} |r_\theta \times r_\varphi|\,dA$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} |\langle -2\sin\theta\sin\varphi, 2\cos\theta\sin\varphi, 0\rangle \times \langle 2\cos\theta\cos\varphi, 2\sin\theta\cos\varphi, -2\sin\varphi\rangle|\,dA$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 4\sin\varphi\,d\theta\,d\varphi = 8\pi(-\cos\varphi)\Big|_0^{\frac{\pi}{2}} = 8\pi$

11. Use the Lagrange Multiplier method to find the maximum product of three positive numbers whose sum is 100.

$f(x,y,z) = xyz$. $g(x,y,z) = x+y+z$. $k=100$

$\max . \ xyz$

s.t $x+y+z=100$

$x.y.z \geq 0$

By Lagrange multiplier method

$$\begin{cases} yz = \lambda \cdot 1 & ① \\ xz = \lambda \cdot 1 & ② \\ xy = \lambda \cdot 1 & ③ \\ x+y+z = 100 & ④ \end{cases}$$

① and ② $xz = yz \Rightarrow z(x-y) = 0 \Rightarrow x=y$

Since $z=0$ will give $xyz=0$, not max.

② and ③ $xz = xy \Rightarrow x(z-y)=0 \Rightarrow y=z$

Since $x=0$ will give $xyz=0$, not max

Thus $x=y=z$. ⑤

By ④ and ⑤ $x=y=z=\frac{100}{3}$

maximum is $\frac{1000000}{27}$