Math 3298 Exam 1
NAME: **SCORE:**

1. Given three points $A(1,0,1)$, $B(1,2,3)$, $C(0,-1,2)$.

3 (a) Find vectors $\vec{AB}$, $\vec{AC}$, $\vec{BC}$.

$\vec{AB} = \langle 1-1, 2-0, 3-1\rangle = \langle 0, 2, 2\rangle$

$\vec{AC} = \langle 0-1, -1-0, 2-1\rangle = \langle -1, -1, 1\rangle$

$\vec{BC} = \langle 0-1, -1-2, 2-3\rangle = \langle -1, -3, -1\rangle$

6 (b) Find $\vec{AB}+\vec{BC}$, $2\vec{AB}$, $\vec{AB}\cdot\vec{AC}$, and $\vec{AB}\times\vec{AC}$.

$\vec{AB}+\vec{BC} = \vec{AC} = \langle -1, -1, 1\rangle$

$2\vec{AB} = \langle 0, 4, 4\rangle$ $\quad \vec{AB}\cdot\vec{AC} = 0-2+2 = 0$

$\vec{AB}\times\vec{AC} = \langle 2\cdot 1-2\cdot(-1), 2\cdot(-1)-0\cdot 1, 0\cdot(-1)-2(-1)\rangle = \langle 4, -2, 2\rangle$

4 (c) Find length of $\vec{AB}$ and area of the parallelogram spanned by $\vec{AB}$, $\vec{AC}$.

$|\vec{AB}| = \sqrt{0^2+2^2+2^2} = \sqrt{8}$

Area $= |\vec{AB}\times\vec{AC}| = \sqrt{4^2+(-2)^2+2^2} = \sqrt{24}$

5 2. Given vectors $\mathbf{a}, \mathbf{b}$. Find $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$.

Since $\vec{a}\times\vec{b} \perp \vec{a} \Rightarrow (\vec{a}\times\vec{b})\cdot\vec{a} = 0$

4 3. Given two unit vectors $\mathbf{u}, \mathbf{v}$. The angle between them is $\pi/4$. What is $\mathbf{u}\cdot\mathbf{v}$?

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

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8

4. Find the equation of a line passing point $P(1, 2, -3)$ and parallel to another line whose equation is $x = 1 + t,\ y = 3 - 5t,\ z = 9 + 2t$.

dir of the line is $\langle 1, -5, 2\rangle$

point $(1, 2, -3)$.

$$\begin{cases} x = 1 + t \\ y = \cancel{2} - 5t \\ z = -3 + 2t \end{cases} \quad -\infty < t < \infty$$

8

5. Find equation of the plane which pass through point $P(2, 3, 1)$ and parallel to another plane $x + 2y - z = 0$.

normal dir of the known plane is $\langle 1, 2, -1\rangle$

Since the plane is // the known plane, ~~the~~ $\langle 1, 2, -1\rangle$ is also the normal dir of the plane.

$$1\cdot(x-2) + 2(y-3) - 1\cdot(z-1) = 0$$

8

6. Find the tangent plane equation for the surface $x^2 + y^2 - z^2 = 0$ at point $(1, 2, \sqrt{5})$.

$\nabla(x^2+y^2+z^2) = \langle 2x, 2y, 2z\rangle = \langle 2, 4, 2\sqrt{5}\rangle$ at pt $(1, 2, \sqrt{5})$

tangent plane eqn is

$$2(x-1) + 4(y-2) + 2\sqrt{5}(z-\sqrt{5}) = 0$$

8

7. Find the point in which the line $x = 2-t,\ y = 1+3t,\ z = t$ intersects the plane $2x-y+z = 2$.

At the intersection pt,

$$\begin{cases} 2x - y + z = 2 \\ x = 2-t,\ y = 1+3t,\ z = t \end{cases} \Rightarrow 2(2-t) - (1+3t) + t = 2$$

$\Rightarrow -4t = 2-3 \Rightarrow t = \frac{1}{4}$

$\Rightarrow$ intersection pt $(\frac{7}{4}, \frac{7}{4}, \frac{1}{4})$

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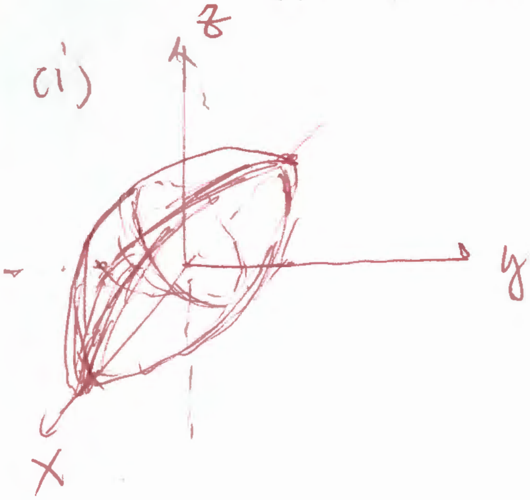
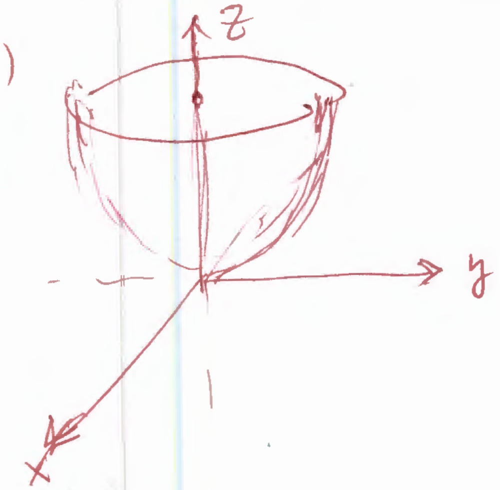
(8)

8. Make a rough sketch of the quadratic surfaces, and give their names (i) $4x^2 + 9y^2 + z^2 = 1$. (ii) $x^2 + 4y^2 - z = 0$.





(8)

9. Find all local max/min for $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

$\begin{cases} f_x = 2x-2 = 0 \\ f_y = 2y-6 = 0 \end{cases} \Rightarrow$ c.p. $(1,3)$

$\begin{cases} 2 = f_{xx} > 0.\quad f_{xy} = f_{yx} = 0 \quad f_{yy} = 2. \\ D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} > 0 \end{cases}$ By 2nd derivative test

Thus $f(1,3)$ is local min $= 4$

(4) 10. Find global max/min value for $f(x, y)$ in last problem on the region E if you already know that the the maximum value and minimum value of $f(x, y)$ on the boundary of E is 24 and 5.

If $(1,3) \in E \Rightarrow$ global max value is 24, min is 4

(1,3) is not on the boundary of E

If $(1,3) \notin E \Rightarrow$ global max value is 24, min is 5

11. Find the directional derivative of the $f(x, y)$ in problem 9 along the direction $\mathbf{u} = \langle 0, 1 \rangle$ at point $(10, 5)$.

(4)

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = f_y(10,5) = 2 \cdot 5 - 6 = 4$

24

6

12. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Find its velocity, speed, and acceleration.

$$\vec{v}(t) = \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad |\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$$

$$\vec{a}(t) = \vec{v}'(t) = \langle -\cos t, -\sin t, 0 \rangle$$

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13. In above problem, find the tangential and normal components of the acceleration a_T and a_N.

$$\vec{T}(t) = \langle -\tfrac{1}{\sqrt{2}}\sin t, \tfrac{1}{\sqrt{2}}\cos t, \tfrac{1}{\sqrt{2}} \rangle$$

$$\vec{T}'(t) = \langle -\tfrac{1}{\sqrt{2}}\cos t, -\tfrac{1}{\sqrt{2}}\sin t, 0 \rangle \Rightarrow |\vec{T}'(t)| = \tfrac{1}{\sqrt{2}}$$

Thus, curvature $= \dfrac{|T'(t)|}{|\vec{r}'(t)|} = \dfrac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \dfrac{1}{2}$

$$\boxed{\begin{cases} a_T = v' = 0 \\ a_N = K|\vec{v}|^2 = \tfrac{1}{2}\cdot(\sqrt{2})^2 = 1 \end{cases}}$$

8

14. A particle starts at the origin with initial velocity $\mathbf{v}(0) = \langle 0, 0, 0 \rangle$. Its acceleration is $\mathbf{a}(t) = \langle 2, 1, 0 \rangle$. Find its trajectory $\mathbf{r}(t)$ and the distance it travels from $t = 0$ to $t = 1$.

$$\vec{v} = \int \vec{a}(t)\,dt + \vec{c} = \langle 2t, t, 0 \rangle + \vec{c}$$

$$\vec{0} = \vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{c} \Rightarrow \vec{c} = \vec{0} \Rightarrow \vec{v} = \langle 2t, t, 0 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2, \tfrac{1}{2}t^2, 0 \rangle + \vec{c}_1$$

$$\vec{0} = \vec{r}(0) = \langle 0, 0, 0 \rangle + \vec{c}_1 \Rightarrow c_1 = 0$$

$$\boxed{\vec{r}(t) = \langle t^2, \tfrac{1}{2}t^2, 0 \rangle}$$

$$s = \int_0^1 |\vec{r}'(t)|\,dt = \int_0^1 \sqrt{(2t)^2 + t^2}\,dt = \int_0^1 \sqrt{5}\,t\,dt = \sqrt{5}\cdot \tfrac{t^2}{2}\Big|_0^1 = \boxed{\tfrac{\sqrt{5}}{2}}$$

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Math 3298 Exam II
NAME: **SCORE:**

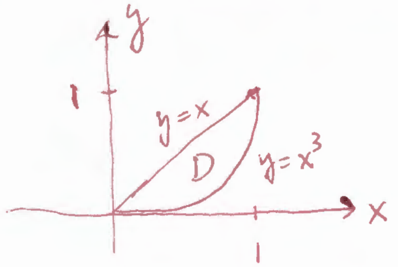
9 1. Evaluate the iterated integral

$$\int_0^1 \int_{x^3}^{x} xy\,dy\,dx.$$

$$= \int_0^1 x \cdot \frac{1}{2}y^2 \Big|_{x^3}^{x} dx = \frac{1}{2}\int_0^1 x(x^2 - x^6)\,dx = \frac{1}{2}\cdot\left(\frac{1}{4}x^4 - \frac{1}{8}x^8\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{8}\right) = \frac{1}{16}$$

9 2. Sketch the region D of the double integral associated with the iterated integral in last problem. Then change of order of the iterated integral to $dxdy$. Evaluate it.



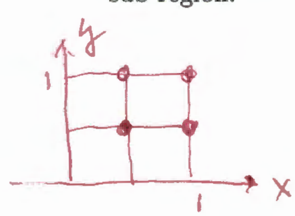
$$\int_0^1 \int_y^{y^{\frac{1}{3}}} xy\,dx\,dy$$

$$= \int_0^1 y \cdot \frac{1}{2}x^2 \Big|_y^{y^{\frac{1}{3}}} dy = \frac{1}{2}\int_0^1 y\left(y^{\frac{2}{3}} - y^2\right)dy$$

$$= \frac{1}{2}\int_0^1 \left(y^{\frac{5}{3}} - y^3\right)dy = \frac{1}{2}\left(\frac{3}{8}y^{\frac{8}{3}} - \frac{1}{4}y^4\right)\Big|_0^1 = \frac{1}{2}\left(\frac{3}{8} - \frac{1}{4}\right)$$

$$= 1/16$$

9 3. Use a Riemann sum with $m = n = 2$ to estimate the double integreal $\iint_D \frac{1}{x^2+y^2}dA$ where $D = [0,1]\times[0,1]$. The sample points are chosen to be the right upper corner of each sub-region.



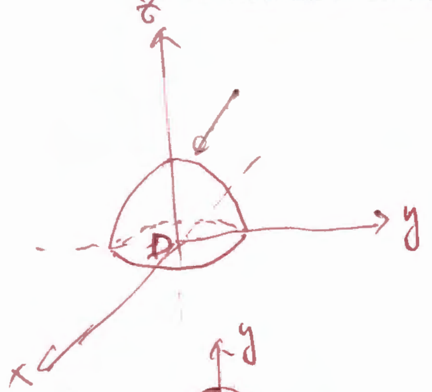
$$\left[f\left(\tfrac{1}{2},1\right) + f(1,1) + f\left(\tfrac{1}{2},\tfrac{1}{2}\right) + f\left(1,\tfrac{1}{2}\right)\right]\Delta A$$

$$= \left(\frac{1}{\frac{1}{4}+1} + \frac{1}{1+1} + \frac{1}{\frac{1}{4}+\frac{1}{4}} + \frac{1}{1+\frac{1}{4}}\right)\cdot\frac{1}{4}$$

$$= \left(\frac{4}{5} + \frac{1}{2} + 2 + \frac{4}{5}\right)\cdot\frac{1}{4} = \frac{41}{40}$$

9 pts

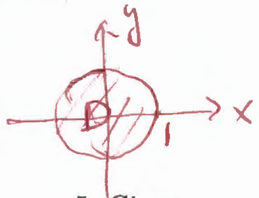
4. Find the surface area of the paraboloid $z = 1 - x^2 - y^2$ that lies above $z = 0$.



$$A(S) = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA$$

$$= \iint_D \sqrt{1+4r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{2}{3}(1+4r^2)^{\frac{3}{2}} \cdot \frac{1}{8}\Big|_0^1 = \frac{\pi}{6}(1+4r^2)^{\frac{3}{2}}\Big|_0^1$$



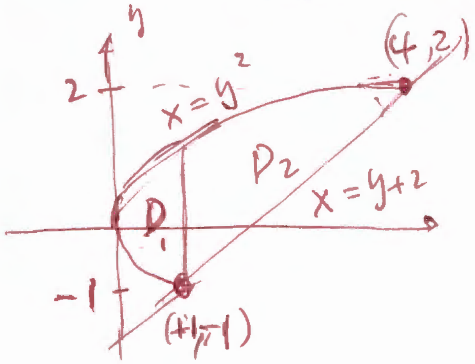
$$= \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$

9 pts

5. Given

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} dydx.$$

Sketch the region of integration. Then change the integration order and evaluate the new integral.



$$= \int_{-1}^{2}\int_{y^2}^{y+2} dx\, dy$$

$$= \int_{-1}^{2} (y+2-y^2)\, dy = \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_{-1}^{2}$$

$$= \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 5 - \frac{1}{2} = \frac{9}{2}$$

9

6. Find the volume of the region below the surface $z = xy$, above $z = 0$, and surrounded by planes $x = 0, x = 1, y = 0, y = 1$.

$$V = \iint_D xy\, dA \qquad D = [0,1] \times [0,1]$$

$$= \int_0^1\int_0^1 xy\, dy\, dx = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

8 pt

7. Use Lagrange multiplier method to find the minimum/maximum of $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

$f_x = y$. $f_y = x$. $g_x = 8x$. $g_y = 2y$

By Lagrange multiplier method

$$\begin{cases} y = \lambda \cdot 8x & ① \\ x = \lambda \cdot 2y & ② \\ 4x^2 + y^2 = 8 & ③ \end{cases}$$

$\Rightarrow$ if $x = 0$; then $y = 0$. ③ is not satisfied

thus. $x \neq 0$. Similarly, if $y = 0 \Rightarrow$ from ②

$x = 0$. again. ③ is violated

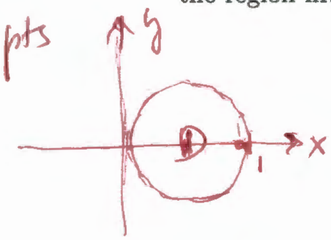
From ① $\lambda = \frac{y}{8x}$.. plug into ② $x = \frac{y}{8x} \cdot 2y \Rightarrow y^2 = 4x^2$

plug into ③ $2y^2 = 8 \Rightarrow y = \pm 2$, $\Rightarrow 4 = 4x^2 \Rightarrow x = \pm 1$

four pts. $(\pm 1, \pm 2)$. $2 = f(1,2) = f(-1,-2)$ max. $f(-1,2) = f(2,-1) = -2$ min

8. Use double integral in polar coordinates to find the area inside the cardioid $r = \cos\theta$. (Sketch the region first to determine the limit of integration)

9 pts



$$A(D) = \iint_D 1 \cdot dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} r\,dr\,d\theta$$

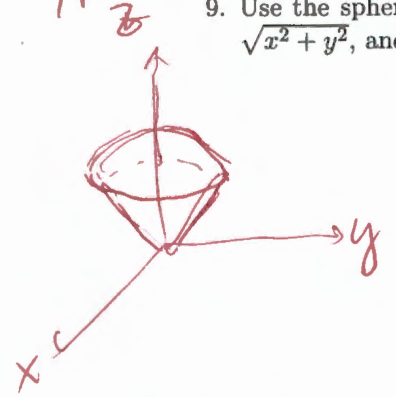
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_0^{\cos\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{4}\left(\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right)$$

$$= \frac{\pi}{4}$$ (D is circle with radius $\frac{1}{2}$)

9 pts

9. Use the spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$, and below the sphere $x^2+y^2+z^2 = 2$.



Cone: $z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2 \Rightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi$

$\Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$

$$V = \iiint_E 1 \cdot dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin\phi\, d\phi \cdot \int_0^{\sqrt{2}} \rho^2 d\rho = 2\pi(-\cos\phi)\Big|_0^{\frac{\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^{\sqrt{2}}$$

$$= 2\pi\left(-\frac{\sqrt{2}}{2}+1\right)\cdot\frac{1}{3}2^{3/2} = \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 2\right)$$

$$= \frac{4\pi}{3}(\sqrt{2}-1)$$

sphere $x^2+y^2+z^2 = 2$

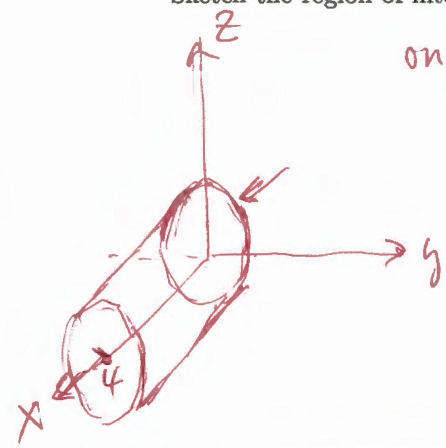
$\Rightarrow \rho^2 = 2$

$\Rightarrow \rho = \sqrt{2}$

5 pts 10. Given the following iterated integrals

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{4} f(x,y,z)\,dx\,dz\,dy.$$

Sketch the region of integration. Then change the integration order to $dydzdx$.

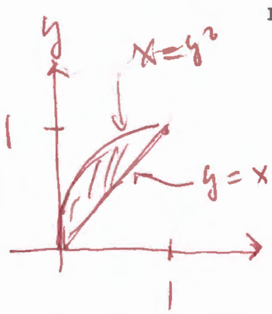


on ZY plane, it is a disk $y^2+z^2 \le 9$

$$= \int_0^4 \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx$$

go y first, the shadow on XZ plane is a rectangle.

11. Find the mass, moments about x, y axis, and center of mass for a lamina that occupies the region D which is bounded by $y = x$ and $x = y^2$. The density is $\rho(x,y) = \sqrt{x}$.



$$m = \iint_D \sqrt{x}\,dA = \int_0^1\int_x^{\sqrt{x}} \sqrt{x}\,dy\,dx = \int_0^1 \sqrt{x}\cdot y\Big|_x^{\sqrt{x}}\,dx$$

$$= \int_0^1 (x - x^{3/2})\,dx = \frac{1}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}}\Big|_0^1 = \frac{1}{2}-\frac{2}{5} = \frac{1}{10}$$

$$M_x = \iint_D \sqrt{x}\cdot y\,dA = \int_0^1\int_x^{\sqrt{x}} \sqrt{x}\cdot y\,dy\,dx = \int_0^1 \sqrt{x}\cdot\frac{1}{2}y^2\Big|_x^{\sqrt{x}}\,dx = \int_0^1 \frac{1}{2}(x^{\frac{3}{2}}-x^{\frac{5}{2}})\,dx$$

$$= \frac{1}{2}\left(\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{7}x^{\frac{7}{2}}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{2}{5}-\frac{2}{7}\right) = \frac{1}{5}-\frac{1}{7} = \frac{2}{35}$$

$$M_y = \iint_D \sqrt{x}\cdot x\,dA = \int_0^1\int_x^{\sqrt{x}} x^{\frac{3}{2}}\,dy\,dx = \int_0^1 x^{\frac{3}{2}}\,y\Big|_x^{\sqrt{x}}\,dx$$

$$= \int_0^1 x^{\frac{3}{2}}(x^{\frac{1}{2}}-x)\,dx = \int_0^1 (x^2 - x^{\frac{5}{2}})\,dx = \frac{1}{3}x^3 - \frac{2}{7}x^{\frac{7}{2}}\Big|_0^1 = \frac{1}{3}-\frac{2}{7}$$

$$= \frac{1}{21}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{21}}{\frac{1}{10}} = \frac{10}{21} \qquad \bar{y} = \frac{M_x}{m} = \frac{\frac{2}{35}}{\frac{1}{10}} = \frac{20}{35} = \frac{4}{7}$$

Math 3298 Exam III
NAME: **SCORE:**

1. Given $\mathbf{F} = \langle xy^2, x^2y \rangle$, and curve C : parabola $y = x^2$, $0 \le x \le 1$. Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

by parameterizing the curve.

$C: \vec{r}(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C xy^2 dx + x^2 y\, dy = \int_0^1 t \cdot (t^2)^2 \cdot dt + (t)^2 t^2 \cdot 2t\, dt$$

$$= \int_0^1 3t^5 dt = \frac{3}{6} t^6 \Big|_0^1 = \frac{1}{2}$$

2. Check that the vector field in problem 1 is conservative. Then compute it by the Fundamental Theorem of Line Integral.

$$\left.\begin{array}{l} P = xy^2 \Rightarrow \frac{\partial P}{\partial y} = 2xy \\ Q = x^2 y \Rightarrow \frac{\partial Q}{\partial x} = 2xy \end{array}\right\} \vec{F} \text{ is conservative}$$

Find $f(x,y)$ s.t. $\nabla f = \vec{F}$. i.e., $\begin{cases} f_x = xy^2 \\ f_y = x^2 y \end{cases}$

$f(x,y) = \frac{1}{2} x^2 y^2$

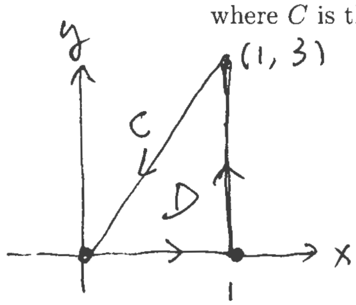
Thus, by the Fundamental Theorem of line integral

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,1) - f(0,0) = \frac{1}{2}$$

3. Use Green's Theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sqrt{1+x^3}dx + 2xydy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ with positive orientation.



$$P(x,y) = \sqrt{1+x^3} \Rightarrow \frac{\partial P}{\partial y} = 0$$

$$Q(x,y) = 2xy \Rightarrow \frac{\partial Q}{\partial x} = 2y$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 2y\, dA$$

$$= \int_0^1 \int_0^{3x} 2y\, dy\, dx = \int_0^1 y^2 \Big|_0^{3x} dx = \int_0^1 9x^2 dx$$

$$= 3x^3 \Big|_0^1 = 3$$

4. Given $\mathbf{F} = \langle e^x + 2xy^2, 2x^2y \rangle$. First, check that $\mathbf{F}$ is conservative. Then, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C : is $\mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t \rangle, 0 \leq t \leq 1$.

$$\left.\begin{array}{l} P = e^x + 2xy^2 \Rightarrow \frac{\partial P}{\partial y} = 4xy \\ Q = 2x^2y \Rightarrow \frac{\partial Q}{\partial x} = 4xy \end{array}\right\} \vec{F} \text{ is conservative}$$

Find $f(x,y)$ s.t. $\nabla f = \vec{F}$, i.e., $\begin{cases} f_x = e^x + 2xy^2 & (1) \\ f_y = 2x^2y & (2) \end{cases}$

By (2) integrate about y, we get $f(x,y) = x^2y^2 + g(x)$ (3)

differentiate (3) about x $\Rightarrow f_x = 2xy^2 + g'(x)$ (4)

Compare (1) and (4) $g'(x) = e^x \Rightarrow g(x) = e^x$ (6)

From (4) and (6) $\boxed{f(x,y) = x^2y^2 + e^x}$

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,1) - f(0,1) = (1+e) - e^0 = e$$

5. Find a parametric equation for the curve $y = \sin x$ starting from $(0,0)$ and ending at $(\pi, 0)$. Specify the parameter region.

$$C: \begin{cases} x = t \\ y = \sin t \end{cases} \qquad 0 \leq t \leq \pi$$

6. Find a parametric equation for the surface which is the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 2$.

$$S: \begin{cases} x = \cos u \\ y = \sin u \\ z = v \end{cases} \qquad (u, v) \in D = [0, 2\pi] \times [0, 2]$$

7. Given a parametric surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, \sin u \rangle, \qquad u \in [-\pi, \pi], \quad v \in [0, 2\pi].$$

Find its Cartesian equation.

$$\begin{cases} x = u \cos v & (1) \\ y = u \sin v & (2) \\ z = \sin u & (3) \end{cases}$$

From (1), (2)

$$x^2 + y^2 = u^2 \qquad (4)$$

$$\Rightarrow \pm \sin\sqrt{x^2+y^2} = \cancel{\pm} \sin u \quad (5)$$

From (3), (5)

$$\boxed{z = \pm \sin\sqrt{x^2+y^2}}$$

for $(x, y) \in D = [-\pi, \pi] \times [-\pi, \pi]$

8. Given a vector field $\mathbf{F} = \langle e^x y^2, e^y z^2, e^z x^2 \rangle$. Find curl$\mathbf{F}$ and div$\mathbf{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \langle 0 - 2e^y z, 0 - 2e^z x, 0 - 2e^x y \rangle$$
$$= -2\langle e^y z, e^z x, e^x y \rangle$$

$$\text{div}\,F = e^x y^2 + e^y z^2 + e^z x^2$$

9. Check whether the vector field

$$\mathbf{F} = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle$$

is conservative. If so, find the potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

$$P = \ln y + 2xy^3 \Rightarrow \frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2$$
$$Q = 3x^2y^2 + \frac{x}{y} \Rightarrow \frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}$$
$> \vec{F}$ conservative

Find $f(x,y)$ s.t. $\nabla f = \vec{F}$

$$\begin{cases} f_x = \ln y + 2xy^3 \quad ① \\ f_y = 3x^2y^2 + \frac{x}{y} \quad ② \end{cases}$$

① $\Rightarrow f(x,y) = \int (\ln y + 2xy^3)\,dx = x\ln y + x^2y^3 + g(y)$

then $f_y = \frac{x}{y} + 3x^2y^2 + g'(y)$ compare with ②

we have $g'(y) = 0 \Rightarrow g(y) = 0$

Therefore $f(x,y) = x\ln y + x^2y^3$

10. Find the surface area of the upper half sphere $x^2 + y^2 + z^2 = 4$ and $z \geq 0$ by surface integral. You need to parameterize the surface first. (hint: using spherical coordinates as parameters is easier)

$$S: \vec{r}(\varphi, \theta) = \langle 2\cos\theta\sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi \rangle$$
$$(\varphi, \theta) \in D = [0, \tfrac{\pi}{2}] \times [0, 2\pi)$$

$$A(S) = \iint_S 1 \cdot dS = \iint_D |r_\varphi \times r_\theta|\, dA = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 4\sin\varphi\, d\varphi\, d\theta$$
$$= 2\pi \cdot 4 \cdot (-\cos\varphi)\Big|_0^{\frac{\pi}{2}}$$
$$= 8\pi$$

$$\vec{r}_\varphi = \langle 2\cos\theta\cos\varphi, 2\sin\theta\cos\varphi, -2\sin\varphi \rangle$$
$$\vec{r}_\theta = \langle -2\sin\theta\sin\varphi, 2\cos\theta\sin\varphi, 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ 2\cos\theta\cos\varphi & 2\sin\theta\cos\varphi & -2\sin\varphi \\ -2\sin\theta\sin\varphi & 2\cos\theta\sin\varphi & 0 \end{vmatrix}$$

$$= \langle 4\cos\theta\sin^2\varphi, 4\sin\theta\sin^2\varphi, 4\cos\varphi\sin\varphi \rangle$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = 4|\sin\varphi|$$