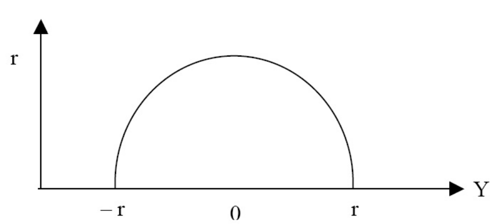
CHAPTER 6

SOME CONTINUOUS PROBABILITY DISTRIBUTIONS

Recall that a *continuous random variable* X is a random variable that takes all values in an interval (or a set of intervals).

- The distribution of a continuous random variable is described by a density function $f(x)$. A density curve must satisfy that
  - The *total area* under the curve, by definition, is equal to **1** or 100%, i.e., $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
  - The *probability* of variable values between a and b is the **area** from a to b under the curve ($a \leq b$), i.e., the area under the curve for a range of values, $\int_a^b f(x)\, dx$, is the *proportion* of all observations for that range.
- The probability of any event is the area under the density curve and above the values of X that make up the event.

EXAMPLE 6.1. What value of r makes the following to be valid density curve?



6.1 Continuous Uniform Distribution

Being the simplest continuous distribution, **uniform distribution** Unif$[A,B]$ is often called "rectangular distribution" because the density function forms a rectangle with base $B-A$ and *constant* height $1/(B-A)$.

Continuous Uniform Distribution

The density function of the continuous uniform random variable X on the interval $[A,B]$ (or, $(A,B]$, $[A,B)$, (A,B)) is

$$f(x;A,B) = \begin{cases} \dfrac{1}{B-A}, & x \in [A,B] \quad \text{(or, } (A,B], [A,B), (A,B)) \\ 0 & \text{elsewhere.} \end{cases}$$

NOTE. The interval may not always be closed. It can be (A,B), $(A,B]$, or $[A,B)$ as well.

Mean and Variance of Continuous Uniform r.v.

The mean and variance of the uniform distribution are

$$\mu = \frac{A+B}{2} \qquad \text{and} \qquad \sigma^2 = \frac{(B-A)^2}{12}.$$

EXAMPLE 6.2. Suppose that X follows the continuous uniform distribution Unif$[2,7]$.

(a) Find the PDF of X. Plot it.

(b) Calculate the mean and the standard deviation of X.

(c) Determine (i) $P(3 \leq X < 6)$, (ii) $P(X \geq 5)$, and (iii) $P(X = 4)$.

(d) Find the value of c such that $P(2 < X \leq c) = 0.4$.

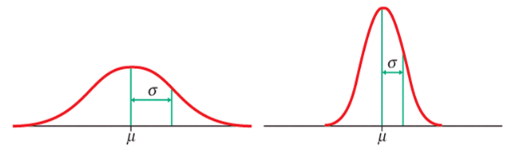
6.2 Normal Distribution

Normal Density Function

The density of the **normal** random variable X with mean μ and variance σ^2 is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

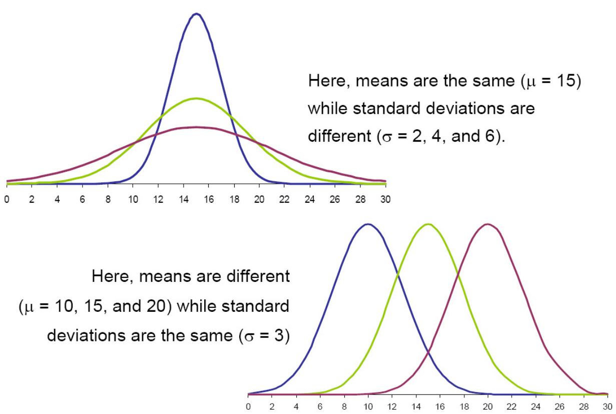
where $e = 2.71828\ldots$ and $\pi = 3.1425926\ldots$.



Normal Density Curve

- a "*bell shaped*" curve.
- depends upon two parameters for its particular shape:
  - μ : the mean of the distribution (i.e. *location*).
  - σ : the standard deviation of the distribution (i.e. *spread* or *variation*).

EXAMPLE 6.3. Given a family of density curves. Which is which?



Properties of a Normal Curve

- The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$.
- The curve is symmetric about a vertical axis through the mean μ.
- The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.
- The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
- The total area under the curve and above the horizontal axis is equal to 1.

EXAMPLE 6.4. Evaluate $\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-1}{3}\right)^2}\, dx$.

Mean and Variance of Normal r.v.

The mean and variance of $\mathrm{n}(x;\mu,\ \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

6.3 Areas under the Normal Curve

Because all normal distributions share the same properties, we can *standardize* our data to transform any normal curve $\mathrm{n}(x;\mu,\sigma)$ into the *standard* normal curve $\mathrm{n}(z;0,\ 1)$ by

$$Z = \frac{X-\mu}{\sigma}.$$

Theorem. If $X \sim \mathrm{n}(x;\ \mu,\sigma)$, then

$$\frac{X-\mu}{\sigma} = Z \ \sim\ \mathrm{n}(z;0,1)$$

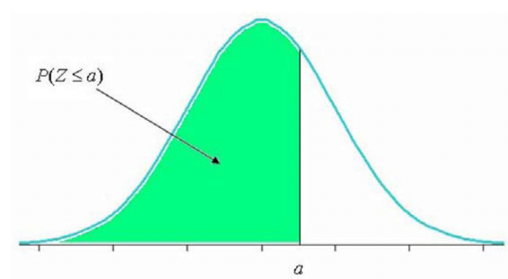
Standard Normal Distribution

The *standard normal distribution* is the normal distribution with **mean 0** and **standard deviation 1**, denoted as $\mathrm{n}(z;0,\ 1)$. Its density function is given by

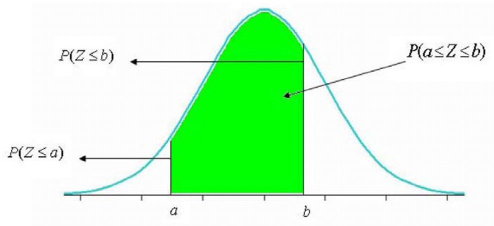
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \qquad -\infty < x < \infty.$$

Suppose that Z follows the standard normal distribution. Then

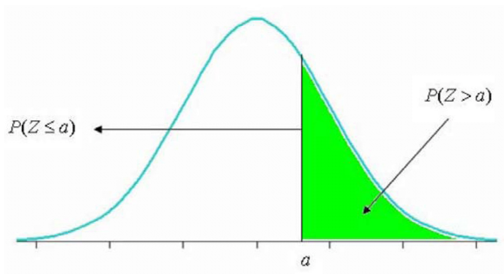
- for $Z \le a$, denoted as $\mathrm{P}(Z \le a)$, is equal to the area under the curve to the left of a.



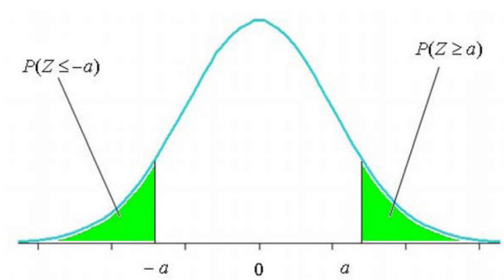
- for $a \le Z \le b$ representing the area under the density curve between a and b, , denoted as $\mathrm{P}(a \le Z \le b)$, is equal to that for $Z \le b$ minus the proportion for $Z \le a$, i.e. $\mathrm{P}(a \le Z \le b) = \mathrm{P}(Z \le b) - \mathrm{P}(Z < a)$.



- for $Z > a$ is equal to 1 minus the proportion for $Z \le a$, i.e. $\mathsf{P}(Z > a) = 1 - \mathsf{P}(Z \le a)$.



- for $Z \ge a$ is equal to that for $Z \le -a$ by symmetry, i.e. $\mathsf{P}(Z \ge a) = \mathsf{P}(Z \le -a)$.



- for $Z = a$ for any a, is equal to 0, i.e. $\mathsf{P}(Z = a) = 0$. It follows that

$$\begin{aligned}\mathsf{P}(a < Z < b) &= \mathsf{P}(a \le Z < b)\\ &= \mathsf{P}(a \le Z \le b)\\ &= \mathsf{P}(a < Z \le b)\end{aligned}$$

EXAMPLE 6.5. Suppose that Z follows the standard normal distribution, i.e. $Z \sim \mathrm{n}(x;\ 0, 1)$. Find

(a) $\mathrm{P}(Z \le 1.05)$

(b) $\mathrm{P}(1.05 \le Z \le 2.38)$

(c) $\mathrm{P}(Z > 1.75)$

(d) $\mathrm{P}(1.05 < Z \le 2.38)$

(e) $\mathrm{P}(1.05 \le Z < 2.38)$

(f) $\mathrm{P}(1.05 < Z < 2.38)$

(g) $\mathrm{P}(-2 < Z \le 1)$

(h) $\mathrm{P}(|Z| \le 1)$

(i) $\mathrm{P}(|Z| \ge 2.45)$

By *standardizing* we convert the desired area for X into an equivalent area associated with Z.

$$X \le a \Longleftrightarrow Z \le \frac{a-\mu}{\sigma}$$

$$a < X \le b \Longleftrightarrow \frac{a-\mu}{\sigma} < Z \le \frac{b-\mu}{\sigma}$$

EXAMPLE 6.6. (a) Suppose that $X \sim \mathrm{n}(x;\ 1, 0.5)$. Find $\mathrm{P}(0 \le X < 1.5)$.

(b) Suppose that $Y \sim \mathrm{n}(y;\ -2, 5)$. Find $\mathrm{P}(Y > 0)$.

Inverse normal calculations

We may also want to find the observed range of values z that correspond to a given probability/ area under the curve. One needs use the normal table backward:

- we first find the desired area/ probability in the body of the table.
- we then read the corresponding z-value from the left column and top row.

For convenience, we shall always choose the z value corresponding to the tabular probability that comes closest to the specified probability.

EXAMPLE 6.7. Let Z be the standard normal random variable. Determine the value of k such that

(a) $\mathrm{P}(Z \le k) = 0.0778$

(b) $\mathrm{P}(-2.88 < Z \le k) = 0.85$

(c) $\mathrm{P}(Z > k) = 0.25$

When dealing with the general normal distribution $X \sim \mathrm{n}(x;\mu,\sigma)$, we still reverse the process and begin with a known area or probability, (i) find the z value, and then (ii) determine x by rearranging the formula

$$z = \frac{x-\mu}{\sigma} \Longleftrightarrow x = \sigma z + \mu.$$

EXAMPLE 6.8. Find the value of k such that

(a) $\mathrm{P}(X \le k) = 0.1977$, where $X \sim \mathrm{n}(x;\ 10, 5)$.

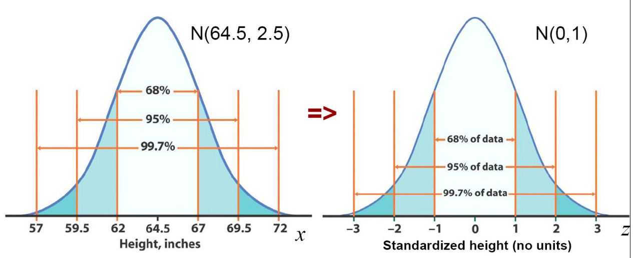
(b) $\mathrm{P}(k < X \le 1.98) = 75\%$, where $X \sim \mathrm{n}(x;\ -1, 2)$.

(c) $\mathrm{P}(|X-2| \le k) = 50\%$, where $X \sim \mathrm{n}(x;\ 2, 1)$.

6.4 Applications of the Normal Distribution

z-score

$z = \frac{x-\mu}{\sigma}$ is often called the z-*score*. It measures the number of standard deviations that a data value x is from the mean μ. When x is larger than the mean μ, z is positive. When x is smaller than the mean μ, z is negative.



EXAMPLE 6.9. Sarah is 22 and her mother Ann is 65 years old. Sarah scores 125 on a standard IQ test and Ann scores 110 on the same test. Scores on this test for the 21-30 age group are approximately normally distributed with mean 110 and standard deviation 25, while scores for the 61-70 age group are approximately normally distributed with mean 90 and standard deviation 25. Who did better?

EXAMPLE 6.10. The length of human pregnancies from conception to birth varies according to a distribution that is approximately normal with mean 266 days and standard deviation 15 days.

(a) What percent of pregnancies last less than 240 days (that's about 8 months)?

(b) What percent of pregnancies last between 240 and 270 days (roughly between 8 months and 9 months)?

(c) How long do the longest 15% of pregnancies last?

EXAMPLE 6.11. The tensile strength of paper used to make grocery bags is a crucial quality characteristic. It is known that the measurement of tensile strength of a type of paper is normally distributed with $\mu = 40\, lb/in^2$ and $\sigma = 2\, lb/in^2$. The purchaser of bags requires to have a strength that is at least $35\, lb/in^2$.

(a) What is the probability that a bag produced using this paper will fail to meet the specification?

(b) Among 10000 bags, how many do you expect failing to meet the specification?

EXAMPLE 6.12. The weights of eggs laid by young hens at a local farm are normally distributed with **unknown** mean μ and standard deviation 5 grams. If approximately 12.1% of eggs weigh less than 45 grams, determine the value of the mean (in grams).

EXAMPLE 6.13. A physical-fitness association is including the mile run in its secondary-school fitness test for boys. The time for this event for boys in secondary school is approximately normally distributed with mean 300 seconds and standard deviation of 20 seconds. If the association wants to designate the fastest 10% as "excellent", what time should the association set for this criterion?

EXAMPLE 6.14. Scores on a provincial math exam are known to be well-described by a normal distribution with mean 600 and standard deviation 100. Exam administrators have decided to give the top 15% of students a grade of A and the next 25% a grade of B. Find the minimum exam score required to receive each of the two letter grades.

6.5 Normal Approximation to the Binomial

Theorem. If $X \sim \mathrm{b}(x;\ n, p)$, then

$$Z = \frac{X-\mu_X}{\sigma_X} = \frac{X-np}{\sqrt{np(1-p)}} \to \mathrm{n}(z; 0, 1),$$

as $n \to \infty$.

NOTE. This approximation works well when n is large and p is not extremely close to 0 or 1. As a rule of thumb, we require both $np > 5$ and $n(1-p) > 5$.

EXAMPLE 6.15. Suppose that X is a binomial random variable with parameters $n = 12$ and $p = 0.4$.

(a) Find $\mathrm{P}(X = 5)$ using the binomial formula.

(b) Find $\mathrm{P}(4.5 < X < 5.5)$ using the normal approximation.

(c) Compare the answers.

EXAMPLE 6.16. Suppose that $X \sim \mathrm{b}(x;\ 12, 0.4)$.

(a) Find $\mathrm{P}(6 \le X \le 8)$ using the binomial formula.

(b) Find $\mathrm{P}(5.5 < X < 8.5)$ using the normal approximation.

(c) Compare the answers.

Continuity Correction

This correction accommodates the fact that a discrete distribution (e.g., binomial) is being approximated by a continuous distribution (e.g., normal). The correction ± 0.5 is called a *continuity correction*.

Normal Approximation to the Binomial Distribution

Let X be a binomial random variable with parameters n and p. For large n, X has approximately a normal distribution with $\mu = np$ and $\sigma^2 = np(1-p)$ and

$$\mathrm{P}(X < x) \approx \mathrm{P}\left(Z < \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(X \le x) \approx \mathrm{P}\left(Z < \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(X > x) \approx \mathrm{P}\left(Z > \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(X \ge x) \approx \mathrm{P}\left(Z > \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

Similarly,

$$\mathrm{P}(a \le X \le b) \approx \mathrm{P}\left(\frac{a - 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{b + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(a < X \le b) \approx \mathrm{P}\left(\frac{a + 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{b + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(a \le X < b) \approx \mathrm{P}\left(\frac{a - 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{b - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$\mathrm{P}(a < X < b) \approx \mathrm{P}\left(\frac{a + 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{b - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

NOTE. Again, this approximation requires $np > 5$ and $n(1-p) > 5$.

EXAMPLE 6.17. A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives

(a) is less than 8?

(b) exceeds 13?

(c) is between 9 and 11 (inclusive)?

Linear Combinations of Normal R.V.'s

Theorem. If $X_1, X_2, \ldots, X_n$ are independent random variables having **normal** distributions with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_2^2$, respectively, then the random variable

$$Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$$

has a **normal** distribution with mean

$$\mu_Y = \mathrm{E}(Y) = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$

and variance

$$\sigma_Y^2 = \mathrm{Var}(Y) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_2^2.$$

In short, if $X_i \overset{\text{independent}}{\sim} \mathrm{N}(\mu_i,\ \sigma_i)$ for $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^n a_iX_i \ \sim\ \mathrm{N}\left(\sum_{i=1}^n a_i\mu_i,\ \sqrt{\sum_{i=1}^n a_i^2\sigma_i^2}\right).$$

6.6 Exponential Distr. and Gamma Distribution

Exponential Distribution

The continuous random variable X has an exponential distribution, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta}e^{-x/\beta}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

where $\beta > 0$.

EXAMPLE 6.18. Find k such that

$$f(x) = \begin{cases} k\ \exp(-2013x), & x > 0 \\ 0, & x \le 0 \end{cases}$$

is a legitimate density function of an exponential r.v..

EXAMPLE 6.19. Evaluate $\int_0^\infty e^{-2x}\,dx$ without using calculus knowledge.

EXAMPLE 6.20. Let X be an exponential random variable with parameter β.

(a) Derive its cumulative density function $F(x)$

(b) Evaluate $\mathrm{P}(0 < X < T)$, where $T > 0$.

(c) Evaluate $\mathrm{P}(X \ge T)$, where $T > 0$.

(d) Find M such that $\mathrm{P}(0 < X < M) = \mathrm{P}(X \ge M)$.

Example 6.21. Suppose that the time, in hours, required to repair a heat pump is a random variable X having an exponential distribution with parameter $\beta = 1/2$.

(a) What is the probability that at most 1 hour will be required to repair the heat pump on the next service call?

(b) What is the probability that at least 2 hours will be required to repair the heat pump on the next service call?

Mean and Variance of the Exponential r.v.

The mean and variance of the *exponential* distribution are

$$\mu = \beta \quad \text{and} \quad \sigma^2 = \beta^2.$$

Example 6.22. Derive the above formulas.

Example 6.23. Refer to Example 6.21. How long do you expect to take to repair a heat pump of this type?

Example 6.24. If an exponential distribution has mean 2.5, what is its standard deviation?

Example 6.25. Evaluate, without using calculus knowledge, the following integrals.

(a) $\displaystyle\int_0^\infty e^{-2x}\,dx$

(b) $\displaystyle\int_0^\infty xe^{-2x}\,dx$

(c) $\displaystyle\int_0^\infty x^2e^{-2x}\,dx$

Let us review the well-known gamma function and some of its important properties.

Gamma Function

The *gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \text{for } \alpha > 0.$$

Properties of the Gamma Function

(a) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

(b) $\Gamma(n) = (n-1)!$, where n is a positive integer.

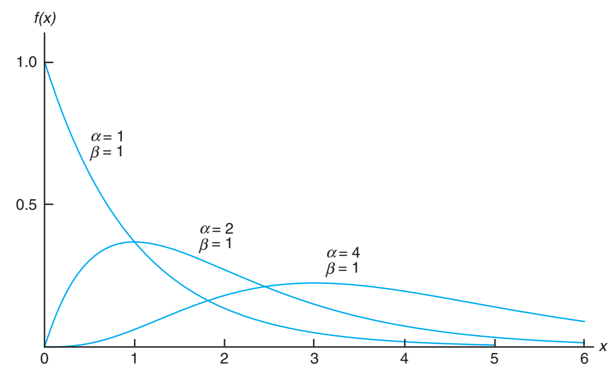
(c) $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$

Gamma Distribution

The continuous random variable X has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

where shape parameter $\alpha > 0$ and rate parameter $\beta > 0$.



Note. The special gamma distribution for which $\alpha = 1$ is called the exponential distribution.

Example 6.26. Evaluate $\displaystyle\int_0^\infty x^{5/2}e^{-x/3}\,dx$.

Mean and Variance of the Gamma r.v.

The mean and variance of the *gamma* distribution are

$$\mu = \alpha\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2.$$

Example 6.27. A biomedical study determines that the survival time, in weeks, has a gamma distribution with $\alpha = 3$ and $\beta = 4$, for a certain dose of a toxicant. What is the probability that a rat survives no longer than 10 weeks?

6.7 Chi-Squared Distribution

The *chi-squared distribution* is another very important special case of the gamma distribution. It is obtained by letting $\alpha = \nu/2$ and $\beta = 2$, where ν is a positive integer and called the *degree of freedom*.

Chi-squared Distribution

The continuous random variable X has a *chi-squared distribution*, with ν *degrees of freedom*, if its density function is given by

$$f(x;\nu) = \begin{cases} \frac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\,x^{\nu/2-1}\,e^{-x/2}, & x > 0 \\ 0, & x \le 0, \end{cases}$$

where ν is a positive integer.

Mean and Variance of the Chi-squared r.v.

The mean and variance of the *chi-squared* distribution are

$$\mu = \nu \quad \text{and} \quad \sigma^2 = 2\nu.$$

Linear Combinations of Chi-squared R.V.'s

Theorem. If $X_1, X_2, \ldots, X_n$ are independent random variables having **Chi-squared** distributions with $\nu_1, \nu_2, \ldots, \nu_n$ degrees of freedom, respectively, then the random variable

$$Y = X_1 + X_2 + \cdots + X_n$$

has a **Chi-squared** distribution with $\nu = \nu_1 + \nu_2 + \cdots + \nu_n$ degrees of freedom.

In short, if $X_i \overset{\text{independent}}{\sim} \chi^2(\nu_i)$ for $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} X_i \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right).$$

Theorem. If $Z \sim \mathrm{N}(0,1)$ then

$$Z^2 \sim \chi^2(1)$$

Theorem. If $X_1, X_2, \ldots, X_n$ are independent and normally distributed with mean μ and standard deviation σ,

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(n).$$

EXAMPLE 6.28. Verify that the exponential distribution with parameter $\beta = 2$ is a chi-squared distribution with parameter $\nu = 2$.

EXAMPLE 6.29. If a ch-squared distribution has mean 2.5, what is its standard deviation?

F distribution

If $F \sim \mathrm{F}(\nu_1, \nu_2)$, then its density is given by

$$h(f) = \frac{\Gamma[(\nu_1+\nu_2)/2](\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \frac{f^{(\nu_1/2)-1}}{(1+\nu_1 f/\nu_2)^{(\nu_1+\nu_2)/2}}$$

Theorem. Let $U \sim \chi^2(\nu_1)$ and $V \sim \chi^2(\nu_2)$. If U and V are independent, then

$$F = \frac{U/\nu_1}{V/\nu_2} \sim \mathrm{F}(\nu_1, \nu_2) \text{ or } \mathrm{F}_{\nu_1,\nu_2}$$

NOTE. Let $\mathrm{F}_\alpha(\nu_1, \nu_2)$ be the upper tail critical value.

$$\mathrm{F}_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{\mathrm{F}_\alpha(\nu_2, \nu_1)}$$

6.8 Beta Distribution

Beta Function

The (complete) *beta function* is defined by

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \text{for } \alpha, \beta > 0, \end{aligned}$$

where $\Gamma(\alpha)$ is the gamma function.

NOTE. The **incomplete beta function** is known as

$$B(x;\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

For $x = 1$, it coincides with the complete beta function.

EXAMPLE 6.30. Define the regularized incomplete beta function in terms of

$$I_x(\alpha,\beta) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}$$

Prove the following properties:

(a) $I_0(\alpha,\beta) = 0$

(b) $I_1(\alpha,\beta) = 1$

(c) $I_x(\alpha,\beta) = I_{1-x}(\beta,\alpha)$

(d) $I_x(\alpha+1,\beta) = I_x(\alpha,\beta) - \dfrac{x^\alpha(1-x)^\beta}{aB(\alpha,\beta)}$

Beta Distribution

The continuous random variable X has a beta distribution, with parameters $\alpha > 0$ and $\beta > 0$, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha, \beta > 0$.

Mean and Variance of the Beta r.v.

The mean and variance of the *beta* distribution are

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

respectively.

EXAMPLE 6.31. Prove that $\mathrm{E}(X) = \dfrac{\alpha}{\alpha+\beta}$, where X has the beta distribution with parameter α and β.

NOTE. The uniform distribution on $(0,1)$ is a beta distribution with parameters $\alpha = 1$ and $\beta = 1$. Its mean and variance are

$$\mu = \frac{1}{1+1} = \frac{1}{2} \quad \text{and} \quad \sigma^2 = \frac{(1)(1)}{(1+1)^2(1+1+1)} = \frac{1}{12}.$$

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