The Family of “Circle Limit III” Escher Patterns

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A rendition of Escher’s *Circle Limit I*.
A rendition of Escher’s *Circle Limit III*. 
The tessellation \( \{8,3\} \).

In general \( \{m, n\} \) denotes the regular tessellation by regular \( m \)-sided polygons meeting \( n \) at a vertex.

The tessellation is hyperbolic if \( (m - 2)(n - 2) > 4 \).
The tessellation \( \{8,3\} \) underlying *Circle Limit III*. 
A Petrie polygon (green), a hyperbolic line through the midpoints of its edge (blue), and two equidistant curves (red).
A kite tessellation superimposed on the *Circle Limit III* pattern.
A *kite* is a quadrilateral $PRQR'$ with two pairs of congruent edges $PR$, $PR'$, and $QR$, $QR'$ (so $\angle PRQ \cong \angle PR'Q$).
A kite is the fundamental region for a tessellation if the angles at P, Q, and R are $2\pi/p$, $2\pi/q$, and $\pi/r$ respectively, for integers $p, q, r \geq 3$. Such a kite tessel-
lation is hyperbolic if $2\pi/p + \pi/r + 2\pi/q + \pi/r < 2\pi$, i.e. if $1/p + 1/q + 1/r < 1$. 
Now $r$ must be odd to make the fish swim head-to-tail, giving a kite tessellation that is the basis for a *Circle Limit III* pattern — which we denote $(p, q, r)$. 
A $(5, 3, 3)$ Circle Limit III pattern.
If $p \neq q$, the backbone lines form equidistant curves, which make an angle $\omega < 90$ degrees with the bounding circle.

For *Circle Limit III*, Coxeter determined that

$$\cos(\omega) = \sqrt{\frac{3\sqrt{2}-4}{8}}, \text{ or } \omega \approx 79.97^\circ.$$  

For the previous pattern, $(5, 3, 3)$, $\cos(\omega) = \sqrt{\frac{3\sqrt{5}-5}{40}}, \text{ or } \omega \approx 78.07^\circ$.

If $p = q$, of course $\omega = 90^\circ$. 
A *Circle Limit I* pattern that solves the “traffic flow” problem.
A recoloring of the previous pattern that gives a uniform color along lines of fish.
A recentering of the previous pattern giving a \((4, 4, 3)\) Circle Limit III pattern.
Future Work

Calculate $\omega$ for arbitrary $(p, q, r)$.

Automatically transform the *Circle Limit III* fish motif to any $(p, q, r)$.

Automatically compute the minimal coloring of a $(p, q, r)$ pattern that satisfies the *Circle Limit III* conditions.