1. The English alphabet contains 21 consonants and five vowels. How many strings of four lowercase letters of the English alphabet contain:
   (a) A vowel in position 2?
   (b) Vowels in positions 2 and 3?
   (c) At least one consonant?
   (d) No vowels in consecutive positions? (‘have’ is OK, ‘beat’ is not)

2. A bitstring question.
   (a) How many bitstrings of length 10 are there?
   (b) How many bitstrings of length 10 contain exactly 3 ones?
   (c) How many bitstrings of length 10 contain exactly 3 ones if exactly one of those ones is in an even position?
   (d) How many bitstrings of length 10 contain exactly 3 ones if none of the ones are in consecutive positions?

3. Arrangements of the word **RECURRANCE**:
   (a) How many arrangements are there?
   (b) How many arrangements have the three R’s in consecutive positions?
   (c) How many arrangements have no consecutive R’s?
   (d) How many arrangements have exactly two consecutive R’s?

4. Some binomial theorem questions:
   (a) Find the coefficient of $x^{17}$ in $(2x - 3)^{40}$.
   (b) Evaluate $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 5^k$.
   (c) Evaluate $\sum_{k=0}^{n} \binom{n}{k} k$.
   (d) Evaluate $\sum_{k=0}^{n} \binom{n}{k} 2^k$.
   (e) Evaluate $\sum_{k=0}^{n} \binom{n}{k} k(-2)^k$.

5. Some pigeonhole problems:
   (a) Show that any subset of $n + 1$ integers between 2 and $2n$ (where $n \geq 2$) always has a pair of integers with no common divisors.
   (b) In a party of 20 people, suppose that there are exactly 48 pairs of people who know each other. Show that someone has 4 or fewer acquaintances.
   (c) Show that any set of $n$ integers with $n \geq 3$ has a pair of integers whose difference is divisible by $n - 1$.
   (d) If the numbers 1 through 10 are arranged randomly in a circle, show that there are three in a row that sum to at least 17.
   (e) If there are 100 people at a party and each person knows an even number of people (possibly zero) show that there are three people with the same number of acquaintances.
6. Show that the sequence \( \binom{n}{0}, 5\binom{n}{1}, 5^2\binom{n}{2}, \ldots \) is unimodal and find the largest term.

7. Give a combinatorial proof for each of the following.
   (a) \( n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1 \) Hint: Count strings \( xyz \) where each of \( x, y, z \) is an integer between 1 and \( n \). For the right hand side, break into cases based on how many 1’s there are in the string.
   (b) \( \binom{3n}{2} = \binom{n}{2} + 2n^2 + \binom{2n}{2} \) Hint: Pick two numbers between 1 and 3\( n \). For the right hand side, use cases based on how many of the numbers are divisible by 3. If you like committee models, you could count how many committees of size 2 could be formed from \( n \) cats and 2\( n \) dogs.
   (c) \( \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n \). One way: count strings of length \( n \) made of \( a \)’s, \( b \)’s, \( c \)’s in two ways. One way: based on how many \( a \)’s the string contains.
   (d) \( \sum_{k=0}^{n} k \binom{n}{k} 2^k = 2n3^{n-1} \). One way: count strings of length \( n \) made of \( a \)’s, \( b \)’s, \( c \)’s in two ways. One way: based on how many \( a \)’s the string contains.

8. Use binomial coefficient identities to solve the following problems.
   (a) Use binomial coefficients to find a formula for \( 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) \).
   (b) Use binomial coefficients to find a formula for \( 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) \).
   (c) Write \( \binom{10}{0}\binom{5}{0} + \binom{10}{1}\binom{5}{1} + \binom{10}{2}\binom{5}{2} + \binom{10}{3}\binom{5}{3} + \binom{10}{4}\binom{5}{4} + \binom{10}{5}\binom{5}{5} \) as a single binomial coefficient. Generalize for \( \binom{2n}{0}\binom{n}{0} + \cdots + \binom{2n}{n}\binom{n}{n} \).
   (d) Evaluate \( \sum_{k=0}^{n} k^2 \binom{n}{k} \).