Math 5366  Some Review Problems
Fall 2015  for Exam 2

1. Find the coefficient of $x^{15}$ in each of the following:

(a) $\frac{1}{(1 - x)^6}$
(b) $\frac{(1 + x^2)^3}{(1 - x)^6}$
(c) $15!(1 + x^2)^3e^{6x}$

2. Find a closed form expression for the generating function for the sequence:

(a) 5, 5, 5, 5, ⋅ ⋅ ⋅
(b) 5, 0, 5, 0, 5, 0, ⋅ ⋅ ⋅
(c) 1, 6, 17, 34, ⋅ ⋅ ⋅ (the $k$’th term is $1 + 2k + 3k^2$)

3. Use the answer to part (c) above to find a formula for $1 + 6 + ⋅ ⋅ ⋅ + (1 + 2n + 3n^2)$.

4. Find the generating function for each of the following problems. Write your answer as simply as possible (that is, $1 + x^2 + x^3 + x^4 + ⋅ ⋅ ⋅$ should be written $\frac{1}{1-x} - x$ or $\frac{1-x+x^2}{1-x}$). Note: Just find the generating function (ordinary or exponential,) don’t solve for $x^n$.

(a) In how many ways can $n$ balls be put into 5 boxes if there is at least one ball in the first box and at most 3 in the second?
(b) How many sequences made up of $a, b, c, d, e$ of length $n$ are there if there is at least one $a$ and at most 3 $b$’s?
(c) How many ways can $n$ balls be put into 5 boxes if the first box has an even number of balls and the second, an odd number?
(d) How many sequences made up of $a, b, c, d, e$ of length $n$ are there if there must be an even number of $a$’s and an odd number of $b$’s?
(e) How many ways can $n$ balls be put into 5 boxes if no box has exactly 2 balls?
(f) How many sequences made up of $a, b, c, d, e$ of length $n$ are there if you can’t have exactly 2 of any letter?

5. Find the solutions to the problems in question 1.

6. Use generating functions to find the number of ways to put $n$ balls into 3 boxes if the first two boxes can hold at most 10 balls. Your answer will vary depending on the size of $n$. For large $n$, simplify your answer as much as possible. Can you prove your answer (for large $n$) by an easy counting argument?
7. Find recurrences, with initial conditions, for each of the following problems.

(a) How many \(n\)-digit ternary sequences do not contain consecutive 0’s?
(b) How many \(n\)-digit ternary sequences **contain** consecutive 0’s?

8. Last time I taught this course, I put the following problem on Exam 2, but the class found it fairly hard (under exam conditions). Suppose we form a sequence out of the numbers 0, 1, 2, 3, 4, 5.

(a) Find a recurrence for \(a_n\), the number of such sequences of length \(n\) which don’t contain consecutive numbers not divisible by 3. That is, sequences like 10403 are ok but 13423 is not because 4 and 2 are consecutive numbers not divisible by 3.
(b) Find the initial conditions for \(a_n\).
(c) Solve the recurrence. There should be fractions but no radicals.
(d) Find a recurrence for the number of such sequences of length \(n\) which DO contain consecutive numbers not divisible by 3. Now 10403 is not ok but 13423 is.

9. Solve the following recurrence relations.

(a) \(a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}\), \(a_0 = 1\), \(a_1 = 4\), \(a_2 = 14\).
(b) \(a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3} + 3 \cdot 4^{n-2}\), \(a_0 = 1\), \(a_1 = -2\), \(a_2 = 6\).
(c) \(a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}\), \(a_0 = 1\), \(a_1 = 2\), \(a_2 = 6\).
(d) \(a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} - 2^n\), \(a_0 = 4\), \(a_1 = 7\), \(a_2 = 21\).
(e) \(a_n = 2a_{n-1} - a_{n-2}\), \(a_0 = 1\), \(a_1 = 3\), USING generating functions.
(f) \(a_n = 6a_{n-1} - 9a_{n-2} + 2 \cdot 3^n\), \(a_0 = 1\), \(a_1 = 0\), with and without generating functions.

10. Solve each of the following recurrence relations.

(a) \(a_n = a_{n-1} + 4a_{n-2} + 4a_{n-3} + \cdots + 4a_0\), \(a_0 = 1\), \(a_1 = 10\).
(b) \(a_n = a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0\), \(a_0 = 1\).