1. Let $S = \{1, 2, 3, \ldots, 50\}$.
   (a) How many subsets does $S$ have?
   (b) How many of size 2?
   (c) How many subsets of size 2 does $S$ have if elements must be at least 25 apart? That is, $\{5, 30\}$ is ok but $\{10, 20\}$ is not.

2. Consider the string YABADABADOO.
   (a) How many arrangements of the string are there?
   (b) How many have all the A’s together?
   (c) How many have all the A’s separated?
   (d) How many don’t have an A in first position, don’t have a B in second position, and don’t have a D in third position?

3. Consider putting $n$ balls into 4 different boxes.
   (a) In how many ways can this be done?
   (b) In how many ways can this be done if the first box has exactly $k$ balls?
   (c) The first box must have some number of balls, $k$, with $0 \leq k \leq n$. Use this and your formula in part (b) to get a combinatorial identity.
   (d) How many ways are there to put $n$ balls into 4 different boxes if no box can have more than 10 things in it? Assume that $n > 40$, and solve the problem in three ways: (1) There is an “obvious” answer, and there is a complicated answer which you can get using (2) generating functions, or (3) using inclusion/exclusion. Comparing the obvious answer to the more complicated answer gives a combinatorial identity.

4. Find the generating function for each of the following sequences.
   (a) $a_n = 7^n$.
   (b) $a_n = n \cdot 7^n$.
   (c) $a_n = \binom{n + 3}{3} 7^n$.
   (d) $a_n = \frac{7^n}{n!}$.
   (e) $a_n = 1 + 7 + 7^2 + \cdots + 7^n$.
   (f) $a_n = 7^{n-1} + 2 \cdot 7^{n-2} + 3 \cdot 7^{n-3} + \cdots + n$. 
5. Find the coefficient of \( x^n \) in each of the following.

(a) \[ \frac{1}{1 - 5x} \]

(b) \[ \frac{x^3}{1 - 5x} \]

(c) \[ \frac{1 + 7x}{(1 - x)(1 - 5x)} \]

(d) \[ (e^{2x} - e^x - x)^2 \]

6. Find a recurrence (with initial conditions) for the number of sequences of 0, 1, \ldots, 6 of length \( n \) which:

(a) do not have consecutive even numbers

(b) have consecutive odd numbers

7. Solve each of the following recurrence relations, both with and without using generating functions.

(a) \[ a_n = 4a_{n-1} - 4a_{n-2} + 2^n, \; a_0 = 2, \; a_1 = 4. \]

(b) \[ a_n = 3a_{n-1} + 4^n, \; a_0 = 0. \]

(c) \[ a_n = na_{n-1} + 2n - 2, \; a_0 = 1. \]

8. Find homogeneous recurrence relations that have the same solutions as in 7a, 7b.

9. (a) How many 10-digit quaternary sequences are there?

(b) How many 10-digit quaternary sequences contain exactly one 0?

(c) How many 10-digit quaternary sequences contain at least one 0?

(d) How many 10-digit quaternary sequences contain exactly two 0’s?

(e) How many 10-digit quaternary sequences contain at least two 0’s?

(f) How many 10-digit quaternary sequences contain an even number of 0’s, an odd number of 1’s, at least one 2 and at least one 3?

(g) How many \( n \)-digit quaternary sequences contain at least one 0, at least one 1 and at least one 3? Solve this by inclusion/exclusion.
10. Some inclusion/exclusion problems.
   (a) How many numbers $k$ with $1 \leq k \leq 1000$ are divisible by 2, 3, or 5?
   (b) How many numbers $k$ with $1 \leq k \leq 10,000$ are NOT divisible by 2, 3 or 5?
   (c) How many solutions to $w + x + y + z = 40$ satisfy $0 \leq w, x \leq 10$ and $0 \leq y, z \leq 15$?
   (d) How many permutations of the letters of the alphabet do not have a vowel in its natural position? (That is, a should not be in position 1, e should not be in position 5, and so on.)

11. Find the rook polynomial for each of the following boards.

   (a) ![Board A](#)
   (b) ![Board B](#)
   (c) ![Board C](#)

12. Suppose the third board in question 11 represents the forbidden positions for a restricted permutation problem:

   ![Forbidden Positions](#)

   Find the number of allowable permutations in two ways:
   (a) By finding the rook polynomial for the associated board. The answer should be the coefficient of $x^4$.
   (b) By using the inclusion/exclusion formula with the restricted position board.