A note on proving things: To prove $\exists x \ P(x)$, one need only give an example of an $x$ for which $P(x)$ is true. Similarly, to demonstrate that $\forall x \ P(x)$ is false, an example of an $x$ making $P(x)$ false is all that is needed. Such an $x$ is called a counter example. Proving $\exists x \ P(x)$ is false or $\forall x \ P(x)$ is true is much harder. We will come back to these later.

In mathematics, we usually have lots of variables floting around. For example, from Section 1.4, we learned that $A \mathbf{x} = \mathbf{b}$ is always consistent if $A$ has a pivot in every row. Here we seem to be swimming in variables: Matrices $A$, vectors $\mathbf{b}$, the rows of $A$, and a hidden one: $A \mathbf{x} = \mathbf{b}$ is consistent means that there is a vector $\mathbf{x}_0$ with the property that when you multiply it by $A$, you get $\mathbf{b}$. All this without even mentioning the constituents of $\mathbf{x}$, that we might typically refer to as variables. How do we handle this kind of thing? We must quantify every variable that appears in a propositional function.

To back up to something simpler, consider the statement

$$(x + 1)^2 = x^2 + 2x + 1.$$ 

What we mean by this is that $\forall x \ ((x + 1)^2 = x^2 + 2x + 1)$. That is, the original statement is not a propostion, but a propositional function. To make it a proposition, we quantify the variable $x$. Extending this,

$$(x + y)^2 = x^2 + 2xy + y^2$$

means

$$\forall x \ \forall y \ ((x + y)^2 = x^2 + 2xy + y^2).$$

We quantify each of the two variables. We might read the above as “For every $x$ and every $y$, ...”

Often, a mathematical statement does not sound like a multivariable quantification, but the only reasonable way to think about it is in those terms. For example, if someone tells you that a positive number times a negative number is negative, what they almost certainly mean is the following:
\[ \forall x \forall y ((x > 0 \land y < 0) \rightarrow xy < 0). \]

That is, a universal claim was being made. In such cases, we say the quantifiers are **nested**. Here is another: If a prime is one more than a multiple of 4, then it is a sum of two squares. We might write this:

\[ \forall p (\exists k (p = 4k + 1) \rightarrow \exists m \exists n (p = m^2 + n^2)). \]

Here, we quantify over all primes \( p \). The first statement inside the \( \forall p (\ ) \) deals with the property that \( p \) is one more than a multiple of 4. The conclusion is that there are integers floating around with \( p \) being the sum of their squares.

Closer to this course, what does it mean to say that you can row reduce a matrix? It means for every matrix \( A \) there is a matrix \( R \) with the property that \( R \) is in reduced echelon form and \( A \) is row equivalent to \( R \). (Slightly) more symbolic, \( \forall A \exists R \) (\( R \) is reduced \land \( A \) is row equivalent to \( R \)).

When you have several variables, order matters. That is, consider the following: (a) \( \forall x \exists y (x + y > 0) \) and (b) \( \exists y \forall x (x + y > 0) \). These are not the same. In fact, (a) is true and (b) is false. Intuitively, you might think of working from the outside in. That is, in (a), \( x \) comes first, and then \( y \) gets its chance. That is, when picking \( y \), you can assume \( x \) is known. However in (b), \( y \) comes first. Given some value of \( y \), is \( x + y > 0 \) true for all \( x \)?

Technically, the way something like \( \forall x \exists y (x + y > 0) \) works is that we view \( \exists y (x + y > 0) \) as a propositional function in \( x \), the variable with no quantifier yet. That is to say, let \( P(x) \) be the propositional function \( \exists y (x + y > 0) \). Then \( P(1) \) is the proposition \( \exists y (1 + y > 0) \), \( P(-10) \) is the propositional function \( \exists y (-10 + y > 0) \), and so on. We see that these are true, and in general, \( \forall x P(x) \) is true.

With \( \exists y \forall x (x + y > 0) \), we first consider the propositional function \( Q(y) \): \( \forall x (x + y > 0) \). Here, \( Q(1) \) says \( \forall x (x + 1 > 0) \), \( Q(-4) \) says \( \forall x (x - 4 > 0) \). These are both false. In particular, \( Q(y) \) is false for all \( y \), so \( \exists y Q(y) \) is false.
How do we negate nested quantifiers? In a fashion similar to ordinary quantifiers: the negation changes $\forall$ to $\exists$, $\exists$ to $\forall$ and then moves inside one level. For example,

$$\neg(\forall x \exists y \,(x + y > 0)) = \exists x \neg(\exists y \,(x + y > 0))$$

$$= \exists x \forall y \neg(x + y > 0)$$

$$= \exists x \forall y \,(x + y \leq 0).$$

For a more complicated example,

$$\neg(\forall p \,(\exists k \,(p = 4k + 1) \rightarrow \exists m \exists n \,(p = m^2 + n^2)))$$

$$= \exists p \neg((\exists k \,(p = 4k + 1) \rightarrow \exists m \exists n \,(p = m^2 + n^2)))$$

$$= \exists p ((\exists k \,(p = 4k + 1) \land \neg(\exists m \exists n \,(p = m^2 + n^2))))$$

$$= \exists p ((\exists k \,(p = 4k + 1) \land \forall m \forall n \neg(p = m^2 + n^2)))$$

$$= \exists p ((\exists k \,(p = 4k + 1) \land \forall m \forall n \,(p \neq m^2 + n^2))).$$

Here, the meaning is that the original claim about primes is wrong if there is a prime which IS one more than a multiple of 4, but which is never the sum of two squares.