Self-assessment questions to accompany Problem Set 4 – Answers

Section 4.3 Laplace Transform Solutions of Linear Differential Equations

Q4.3.1 Laplace transforms can be used to solve linear differential equations by taking the Laplace transform of each term in the differential equation. Next, the equation is algebraically solved for the dependent variable in the transform space. That is, the Laplace transform of the dependent variable, \( F(s) \), is expressed as a function of \( s \) and the initial conditions. Finally, the inverse Laplace transform is applied to the previous equation, on the left hand side of the equation, \( F(s) \) becomes \( f(t) \) and the inverse Laplace transform of the right hand side of the equation is the analytical expression of \( f(t) \) in terms of \( t \) which represents the time solution of the original differential equation.

Q4.3.2 The primary advantage of using Laplace transforms to solve linear differential equations is that this method is relatively easy to implement, i.e., each step requires standard procedures that can be directly applied. In general, Laplace transforms use algebraic equations to solve differential equations.

Q4.3.3 After Laplace transforms have been applied to a linear differential equation, the equation for \( F(s) \) must be converted into a series of terms that allow the application of inverse Laplace transforms. That is, for example \( F(s) \) must be converted into terms consistent with the Laplace transforms listed in Table 4.1.

P4.3.1 Taking the Laplace transform of each term in the differential equation yields \( sY(s) - y_0 = -Y(s) \). Solving for \( Y(s) \) yields \( Y(s) = \frac{y_0}{s+1} \). Applying an inverse Laplace transform yields \( y(t) = y_0 e^t \).

Section 4.4 Individual Real Poles

Q4.4.1 When a Laplace transform is expressed as \( F(s) = \frac{N(s)}{D(s)} \), the value of \( s \) that makes \( D(s) = 0 \) (roots) are the poles of \( F(s) \).

Q4.4.2 Partial fractions are applied to Laplace transforms to put the Laplace transform into a form that can be conveniently transformed to the time domain.

Q4.4.3 Partial fraction expansions are formed by factoring the denominator of a Laplace transform into its roots (poles of the Laplace transform) and expressing the Laplace transform as a series of terms each of which contains one of the roots in its denominator:

\[
Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)} = \frac{C_1}{s+a_1} + \frac{C_2}{s+a_2} + \cdots + \frac{C_n}{s+a_n}
\]

Q4.4.4 The values of the real poles of a Laplace transform are converted into exponential terms when the Laplace transform is converted to the time domain. That is, if \( a \) is a pole, when \( F(s) \) converted to the time domain, the time-domain solution will contain an \( e^{at} \) term.

P4.4.1 Factoring the denominator and applying a partial fraction expansion yields

\[
Y(s) = \frac{1}{s^2 + 5s + 6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}
\]

Applying the Heaviside method yields

\[
C_1 = \left. \frac{N(s)}{D(s)} \right|_{s=-2} = \frac{1}{s+3} \bigg|_{s=-2} = 1 \quad C_2 = \left. \frac{N(s)}{D(s)} \right|_{s=-3} = \frac{1}{s+2} \bigg|_{s=-3} = -1
\]

Finally,

\[
Y(s) = \frac{1}{s+2} - \frac{1}{s+3}
\]

P4.4.2 Applying inverse Laplace transforms (Table 4.1) to the result from P4.4.1 yields \( y(t) = e^{-2t} - e^{-3t} \)

Question 2: 0-1023

Question 3: 1010, 10