Ch 5 Probability: The Mathematics of Randomness
5.1.1 Random Variables and Their Distributions

• A random variable is a quantity that (prior to observation) can be thought of as dependent on chance phenomena.

Toss a coin 10 times
X=# of heads

Toss a coin until a head
X=# of tosses needed
More random variables

Toss a die
   \( X = \) points showing

Plant 100 seeds of pumpkins
   \( X = \%) \) germinating

Test a light bulb
   \( X = \) lifetime of bulb

Test 20 light bulbs
   \( X = \) average lifetime of bulbs
Types of Random Variable

• A discrete random variable is one that has isolated or separated possible values.  
  (Counts, finite-possible values)

• A continuous random variable is one that can be idealized as having an entire (continuous) interval of numbers as its set of possible values.  
  (Lifetimes, time, compression strength $0 \rightarrow \infty$)
Probability Distribution

• To specify a probability distribution for a random variable is to give its set of possible values and (in one way or another) consistently assign numbers between 0 and 1—called probabilities— as measures of the likelihood that be various numerical values will occur.

• It is basically a rule of assigning probabilities to random events.

Roll a die, $X = \#$ showing

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
</tr>
</tbody>
</table>
Probability Distribution

The values of a probability distribution must be numbers on the interval from 0 to 1.

The sum of all the values of a probability distribution must be equal to 1.
Example

Inspect 3 parts. Let $X$ be the # of parts with defect.

Find the probability distribution of $X$.

First, what are possible values of $X$?

$(0, 1, 2, 3)$
When does $X$ take on each value?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>{NNN}</td>
<td>0</td>
</tr>
<tr>
<td>{NND, NDN, DNN}</td>
<td>1</td>
</tr>
<tr>
<td>{NDD, DND, DDN}</td>
<td>2</td>
</tr>
<tr>
<td>{DDD}</td>
<td>3</td>
</tr>
</tbody>
</table>

- If $P(D)=0.5$, then $P(N)=0.5$  We have

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)=P(X=x)$</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>
Cumulative Distribution Function

• **Cumulative distribution** $F(x)$ of a random variable $X$ is

\[ F(x) = P(X \leq x) \]

• Since (for discrete distributions) probabilities are calculated by summing values of $f(x)$, for a discrete distribution

\[ F(x) = P(X \leq x) \]
“Defect” Example continued

We had

\[ f(0) = \frac{1}{8}, \quad f(1) = f(2) = \frac{3}{8}, \quad f(3) = \frac{1}{8}. \]

Therefore

\[
F(x) = \begin{cases} 
0, & x < 0; \\
\frac{1}{8}, & 0 \leq x < 1; \\
\frac{1}{2}, & 1 \leq x < 2; \\
\frac{7}{8}, & 2 \leq x < 3; \\
1, & x \geq 3.
\end{cases}
\]

• \( F(0) = f(0) = \frac{1}{8}; \)
• \( F(1) = f(0) + f(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}; \)
• \( F(2) = f(0) + f(1) + f(2) = \frac{7}{8}; \)
• \( F(3) = f(0) + f(1) + f(2) + f(3) = 1. \)
Summaries of Discrete Probability Distributions

Given a set of numbers, for example,

\[ S = \{1, 1, 1, 1, 3, 3, 4, 5, 5, 6\} \]

To calculate the average of these 10 numbers, you can use

\[
\frac{1 + 1 + 1 + 1 + 3 + 3 + 4 + 5 + 5 + 6}{10} = \frac{30}{10} = 3
\]

Or you can get it this way

\[
\frac{(1)(4) + (3)(2) + (4)(1) + (5)(2) + (6)(1)}{10} = (1)\left(\frac{4}{10}\right) + (3)\left(\frac{2}{10}\right) + (4)\left(\frac{1}{10}\right) + (5)\left(\frac{2}{10}\right) + (6)\left(\frac{1}{10}\right) = 3.
\]
Mean or Expectation

The above result is the weighted sum of $x$ values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>4/10</td>
<td>2/10</td>
<td>1/10</td>
<td>2/10</td>
<td>1/10</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The weights are actually the probability distribution for a random variable $X$.

The weighted sum, $3$, is called the mean or mathematical expectation of random variable $X$. 
**Definition**

Let $X$ be a discrete random variable with probability distribution $f(x)$. The mean or expected value of $X$ is

$$
\mu = E(X) = \sum_x xf(x)
$$

The **expectation or mean of a random variable** $X$ is the long run average of the observations from the random variable $X$.

The mean of $X$ is not necessarily a possible value for $X$. 
Back to “defect” example

• What is the average number of defective items we expect to see?

<table>
<thead>
<tr>
<th>X</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>3/8</td>
</tr>
<tr>
<td>2</td>
<td>3/8</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
</tr>
</tbody>
</table>

\[
\mu = E(X) = \sum x \cdot f(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}
\]
Variance

The mean is a measure of the center of a random variable or distribution

\[
X: \quad x = -1 \quad 0 \quad 1 \\
\begin{array}{c} f(x) \end{array} = \begin{array}{c} \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \end{array}
\]

\[
Y: \quad y = -2 \quad 0 \quad 2 \\
\begin{array}{c} g(y) \end{array} = \begin{array}{c} \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \end{array}
\]

Both variables have the same center: 0

Which has a larger variability?
Variance

• Deviation from the center (mean)
  \[ X - \mu_X \]
  The average of \( X - \mu_X \) is zero!

• Consider the weighted average of \( (X - \mu_X)^2 \)

\[
E(X-0)^2 = (1)(1/4) + (0)(1/2) + (1)(1/4) = 1/2 \\
E(Y-0)^2 = (4)(1/4) + (0)(1/2) + (4)(1/4) = 2
\]

\( Y \) is more variable!
Definition

- Let $X$ be a discrete random variable with probability distribution $f(x)$ and mean $\mu$.

- The **variance of $X$** is

  $$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x)$$

The positive square root of the $\sigma^2$ is called **standard deviation** of $X$, denoted by $\sigma$. 
Back to “defect” example

<table>
<thead>
<tr>
<th>X</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>3/8</td>
</tr>
<tr>
<td>2</td>
<td>3/8</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
</tr>
</tbody>
</table>

\[
\sigma^2 = Var(X) = E(x - \mu)^2 = \sum (x - \mu)^2 \cdot f(x)
\]

\[
= (0 - 1.5)^2 \times \frac{1}{8} + (1 - 1.5)^2 \times \frac{3}{8} + (2 - 1.5)^2 \times \frac{3}{8} + (3 - 1.5)^2 \times \frac{1}{8}
\]
Alternatively, we can find $\text{Var}(X)$ from $\sigma^2 = E(x - \mu)^2 = E(x^2) - \mu^2$

To show this:

\[
E(X - \mu)^2 = E(X^2 - 2\mu X + \mu^2) \quad \text{average in long run}
\]
\[
= E(X^2) - 2\mu E(X) + E(\mu^2)
\]
\[
= E(X^2) - 2\mu \cdot \mu + \mu^2
\]
\[
= E(X^2) - \mu^2. 
\]

$E(X^2) = \text{weighted average of } X^2$

\[
= \sum x^2 f(x)
\]

\[
E(X^2) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3
\]

\[
\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = 3 - 1.5^2 = 0.75
\]

\[
\sigma = \sqrt{\sigma^2} = \sqrt{0.75}
\]
Binomial Distribution

In many applied problems, we are interested in the probability that an event will occur x times out of n.
For example

Inspect 3 items. $X=$# of defects.
  D=a defect, N=not a defect, Suppose $P(N)=\frac{5}{6}$

No defect: $(x=0)$
  $NNN \rightarrow (\frac{5}{6})(\frac{5}{6})(\frac{5}{6})$

One defect: $(x=1)$
  $NND \rightarrow (\frac{5}{6})(\frac{5}{6})(\frac{1}{6})$
  $NDN \rightarrow $ same
  $DNN \rightarrow $ same

Two defect: $(x=2)$
  $NDD \rightarrow (\frac{5}{6})(\frac{1}{6})(\frac{1}{6})$
  $DND \rightarrow $ same
  $DDN \rightarrow $ same

Three defect: $(x=3)$
  $DDD \rightarrow (\frac{1}{6})(\frac{1}{6})(\frac{1}{6})$
Binomial distribution

- $x$  $f(x)$
  - 0  $(5/6)^3$
  - 1  $3(1/6)(5/6)^2$
  - 2  $3(1/6)^2(5/6)$
  - 3  $(1/6)^3$

$$f(x) = \binom{3}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{3-x}$$

How many ways to choose $x$ of 3 places for D

$[P(D)]^\# \text{ of } D$

$[1-P(D)]^{3-\#} \text{ of } D$
In general:

- n independent trials
- p probability of a success
- x = # of successes

\[
\binom{n}{x} \text{ ways to choose } x \text{ places for success,}
\]

\[
f(x) = \binom{n}{x} p^x (1 - p)^{n-x}
\]
• Roll a die 20 times. X=# of 6’s, n=20, p=1/6

\[ f(x) = \binom{20}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{20-x} \]

\[ p(x = 4) = \binom{20}{4} \left( \frac{1}{6} \right)^4 \left( \frac{5}{6} \right)^{16} \]

• Flip a fair coin 10 times. X=# of heads

\[ f(x) = \binom{10}{x} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{10-x} = \binom{10}{x} \left( \frac{1}{2} \right)^{10} \]
More example

• Pumpkin seeds germinate with probability 0.93. Plant n=50 seeds

• $X =$ # of seeds germinating

\[
f(x) = \binom{50}{x}(0.93)^x(0.07)^{50-x}
\]

\[
P(X = 48) = \binom{50}{48}(0.93)^{48}(0.07)^2
\]
Mean of Binomial Distribution

- \( X \) = \# of defects in 3 items

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((5/6)^3)</td>
</tr>
<tr>
<td>1</td>
<td>(3(1/6)(5/6)^2)</td>
</tr>
<tr>
<td>2</td>
<td>(3(1/6)^2(5/6))</td>
</tr>
<tr>
<td>3</td>
<td>((1/6)^3)</td>
</tr>
</tbody>
</table>

Expected long run average of \( X \)?
The mean of a binomial distribution

• Binomial distribution
  
n= # of trials,
  p=probability of success on each trial
  X= # of successes

\[
E(x) = \mu = \sum x \binom{n}{x} p^x (1 - p)^{n-x} = np
\]
• Let’s check this.

\[ E(X) = \mu_X = \sum x f(x) = \sum_{x=0}^{3} x \binom{3}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{3-x} \]

\[ = 0 \cdot \left( \frac{5}{6} \right)^3 + 1 \cdot 3 \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right) + 2 \cdot 3 \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right) + 3 \cdot \left( \frac{1}{6} \right)^3 = \frac{1}{2} \]

• Use binomial mean formula

\[ E(x) = \mu = \sum x \binom{n}{x} p^x (1 - p)^{n-x} = np = 3 \cdot \frac{1}{6} = \frac{1}{2} \]
More examples

• Toss a die $n=60$ times, $X=$# of 6’s
  known that $p=1/6$
  
  $\mu = \mu_X = E(X) = np = (60)(1/6) = 10$

We expect to get 10 6’s.
Variance for Binomial distribution

- \( \sigma^2 = np(1-p) \)
  where \( n \) is # of trials and \( p \) is probability of a success.
- From the previous example, \( n=3, \ p=1/6 \)
  Then
  \[ \sigma^2 = np(1-p) = 3 \times \frac{1}{6} \times (1-\frac{1}{6}) = \frac{5}{12} \]
Geometric distribution

- Constant probability of success = $p$
- $X = \#$ of trials to first success

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p$</td>
</tr>
<tr>
<td>2</td>
<td>$(1 - p)p$</td>
</tr>
<tr>
<td>3</td>
<td>$(1 - p)(1 - p)p$</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>$(1 - p)^{x-1}p$</td>
</tr>
</tbody>
</table>

$f(x) = P(x) = p(1 - p)^{x-1}$
For $p = 0.1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>$0.9 \times 0.1$</td>
</tr>
<tr>
<td>3</td>
<td>$0.9 \times 0.9 \times 0.1$</td>
</tr>
</tbody>
</table>

$$\mu = E(X) = \sum x \times p(1 - p)^{x-1} = \frac{1}{p}$$
For $p = 0.1$ \hspace{1cm} $E(X) = 10$

$p = \frac{1}{6}$ (Dice doubles) \hspace{1cm} $E(X) = 6$ turns

To find the variance,

$$\sigma^2 = Var(x) = \frac{1-p}{p^2}$$

For $p = 0.1$
$$\sigma^2 = \frac{0.9}{0.1^2} = 90 \hspace{1cm} \sigma = 9.49$$
$$\mu = 10$$
5.1.5 Poisson Distribution

Events happen randomly at a uniform rate over time or space

Radioactive decays in time

Meteor strikes on a map

\[ X = \# \text{ of events in given time or space} \]

\[ \lambda = E(X) \text{ and } \lambda = Var(X) \]

\[ f(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, \ldots \]
Example 6 (p. 242)
\[ x = \alpha \text{ particles from bar of polonium } \lambda = 3.87 \]
\[ P(X = 4) = e^{-3.87} \times 3.87^4 / 4! = 0.1950 \]

Example 7 (p.242) Arrivals at library
If events happen at a uniform rate, then in a time interval, t,
\[ \lambda = \text{expected } \# = \text{rate } \times \text{time} \]
Exercise 9 (p.244) Transmission line interruptions
rate = 1 per day

(a) 5 days $\lambda = 1 \times 5 = 5$
$P(X = 0) = \frac{e^{-5} \times 5^0}{0!} = e^{-5} = 0.0067$

(b) 4 weeks
$Y = \#$ of weeks with no interrupts
$Y \sim$ Binomial $n = 4 \quad p = 0.0067$

$$P(Y = 2) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$

$$= \frac{4!}{2!2!} 0.0067^2 (1 - 0.0067)^{4-2} = 0.000266$$