Weibull exercise

Lifetimes in years of an electric motor have a Weibull distribution with $\beta = 3$ and $\alpha = 5$.

Find a guarantee time, $g$, such that 95% of the motors last more than $g$ years.
That is, find $g$ so that $P(X > g) = 0.95$

What is called a linear combination of random variables?

$aX+b, aX+bY+c$, etc.

Let $U = a_0 + a_1X + a_2Y + a_3Z + \cdots$

Means of Linear Combinations

$E(U) = a_0 + a_1EX + a_2EY + a_3EZ + \cdots$

Example

$X, Y, Z =$ values of three dice
You win $U = 5 + 2X + 3Y + 4Z$ dollars
$E(X) = E(Y) = E(Z) = 3.5$
Your total expected winnings are
$E(U) = 5 + 2(3.5) + 3(3.5) + 4(3.5) = 36.50$

Expected value (constant) = constant

$E(5) = 5$

$E(\sum) = \sum (\text{expected values})$


$E(\text{constant} \cdot X) = \text{constant} \cdot EX$

$E(3X) = 3E(X)$

$E(4Y) = 4E(Y)$

$E(5Z) = 5E(Z)$

$E(a_0 + a_1X + a_2Y + a_3Z)$

$= E(a_0) + E(a_1X) + E(a_2Y) + E(a_3Z)$

$= a_0 + a_1EX + a_2EY + a_3EZ$

Variances of Linear Combinations:

$\text{Var}(a_0 + a_1X + a_2Y + a_3Z)$

$= a_1^2 \text{Var}(X) + a_2^2 \text{Var}(Y) + a_3^2 \text{Var}(Z)$

when $X, Y$ and $Z$ are independent.

$\text{Var}(a_0) = 0$

$\text{Var}(\text{constant}) = 0$

$\text{Var}(a_0 + X) = \text{Var}(X)$

Adding a constant doesn’t change variance.

$\text{Var}(a_0X) = a_1^2 \text{Var}(X)$

Variances are in units$^2$.

$\text{St. Dev}(a_0X) = |a_1| \text{ St. Dev}(X)$
If X, Y, Z are independent or uncorrelated.

\[ \text{Var}(X + Y + Z) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(Z) \]

Putting these facts together

\[ \text{Var}(a_0 + a_1X + a_2Y + a_3Z) = \begin{cases} \text{Var}(a_0) + \text{Var}(a_1X) + \text{Var}(a_2Y) + \text{Var}(a_3Z) & \text{if independent} \\ \end{cases} \]

\[ = a_1^2\text{Var}(X) + a_2^2\text{Var}(Y) + a_3^2\text{Var}(Z) \]

For independent X, Y

\[ \sigma_{X,Y} = \sqrt{\text{Var}(X) + \text{Var}(Y)} \]

Suppose that “20 ohm” resistors have a mean of 20 ohms and a standard deviation of 0.2 ohms. Four independent resistors are to be chosen a connected in series.

- \( R_1 \) = resistance of resistor 1
- \( R_2 \) = resistance of resistor 2
- \( R_3 \) = resistance of resistor 3
- \( R_4 \) = resistance of resistor 4

The system’s resistance, \( S \), is

\[ S = R_1 + R_2 + R_3 + R_4 \]

Find the mean and standard deviation for the system resistance, \( S \).

Most useful facts:

- If \( X_1, X_2, X_3, \ldots, X_n \) are independent measurements each with \( E(X_i) = \mu_i \) and \( \text{Var}(X_i) = \sigma_i^2 \) then
- \( E(X) = \mu \)
- \( \text{Var}(X) = \frac{\sigma^2}{n} \)
- The sample mean \( \bar{X} \) is an unbiased estimator of the population mean \( \mu \).

\[ E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu_i = \mu \]

The mean \( \bar{X} \) is an unbiased estimator of the population mean \( \mu \).

\[ \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n} \]

### 5.5.4 Propagation of Error

- Often done in chemistry or physics labs.
- In last section we talked about the case when \( U = g(x) \) is linear in \( x \)

\[ E\left(\alpha + bX\right) = \alpha + bE(X) \]

\[ g(x) = \alpha + bx, \quad E(g(x)) = g(E(X)) \]

- If \( U = g(X, Y, Z) \) is not linear

\[ \frac{\partial}{\partial x} (x \cdot EX) + \frac{\partial}{\partial y} (y \cdot EY) + \frac{\partial}{\partial z} (z \cdot EZ) \]

Where the partial derivatives are evaluated at \( (x, y, z) = (EX, EY, EZ) \).
The right hand side of the equation is linear in $x, y, z$.

For the mean $E(U)$

$$g(x, y, z) = g(EX, EY, EZ) + \frac{\partial g}{\partial x}(x - EX) + \frac{\partial g}{\partial y}(y - EY) + \ldots + \frac{\partial g}{\partial z}(z - EZ)$$

$$E(U) = E(g(x, y, z)) = E(g(EX, EY, EZ)) + \frac{\partial g}{\partial x}(EX - E) + \frac{\partial g}{\partial y}(EY - E) + \ldots + \frac{\partial g}{\partial z}(EZ - E)$$

since

$$E \left[ \frac{\partial g}{\partial x} (x - EX) \right] = \frac{\partial g}{\partial x} (EX - EX) = \frac{\partial g}{\partial x} \cdot 0 = 0$$

For $\text{Var}(U)$

$$\text{Var}(U) = \text{Var}(g(x, y, z)) = \left( \frac{\partial g}{\partial x} \right)^2 \text{Var}(X) + \left( \frac{\partial g}{\partial y} \right)^2 \text{Var}(Y) + \ldots + \left( \frac{\partial g}{\partial z} \right)^2 \text{Var}(Z)$$

Example 24 on page 311

The assembly resistance $R$ is related to $R_1$, $R_2$ and $R_3$ by

$$R = R_1 + \frac{R_2 - R_1}{2} + \frac{R_3 - R_1}{2}$$

A large lot of resistors is manufactured and has a mean resistance of 100 Ω with a standard deviation of 2 Ω. If 3 resistors are taken at random, can you approximate mean and variance for the resulting assembly resistance?

Example

$$\text{Var}(R) = \left( \frac{1}{100} \right)^2 \text{Var}(\text{Resistor})$$

$E(R) = 100 + \left( \frac{1}{100} \right) \text{Var}(R)$

$$E(R) = 100 + \left( \frac{1}{100} \right) \cdot 4$$

$$E(R) = 100 + \frac{4}{100}$$

$$E(R) = 100 + 0.04$$

$$E(R) = 100.04$$

$\text{Var}(R) = 150$
\[
\text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \text{Var}(X_i) = \frac{1}{n^2} \left( \frac{\sigma^2}{n} \right) = \frac{\sigma^2}{n^3}
\]

5.5.5 The Central Limit Effect

- If \(X_1, X_2, \ldots, X_n\) are independent random variables with mean \(\mu\) and variance \(\sigma^2\), then for large \(n\), the variable \(\bar{X}\) is approximately normally distributed.

Central Limit Theorem

If the population is normal or sample size is large, mean \(\bar{X}\) follows a normal distribution

\[\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})\]

and

\[z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\]

follows a standard normal distribution.

• The closer \(X\)’s distribution is to a normal distribution, the smaller \(n\) can be and have the sample mean nearly normal.

• If \(X\) is normal, then the sample mean is normal for any \(n\), even \(n=1\).

• Usually \(n=30\) is big enough so that the sample mean is approximately normal unless the distribution of \(X\) is very asymmetrical.

• If \(X\) is not normal, there are often better procedures than using a normal approximation, but we won’t study those options.
If $X_1, X_2, \ldots, X_n$ are normal, then $\bar{X}$ is exactly normal.

Example: Bags of potatoes weigh

$\mu = 5.0$ pounds

$\sigma = 0.1$ pounds

If we buy 4 bags, what is $P(\bar{X} < 4.9)$?

$E(\bar{X}) = \mu = 5$  \hspace{1cm} $\text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{0.1^2}{4} = 0.05$

$P(\bar{X} < 4.9) = P \left( \frac{\bar{X} - \mu}{\sqrt{\text{Var}(\bar{X})}} < \frac{4.9 - 5.0}{0.05} \right) = P(z < -2) = 0.0228$

Exercise

The mean of a random sample of size $n=100$ is going to be used to estimate the mean daily milk production of a very large herd of dairy cows. Given that the standard deviation of the population to be sampled is $\sigma=3.6$ quarts, what can we assert about the probabilities that the error of this estimate will be more then 0.72 quart?