Math 3298 Practice Midterm 1 Solutions

Please notify me as soon as possible if you believe there is an error in these solutions.

(1) (a) Show that if $d < 1$ there are no vertical tangents to the curve $x(t) = t - d \sin t$, $y(t) = 1 - d \cos t$.

Solution: Vertical tangents are only possible at points where $dx/dt = 0$. Since $dx/dt = 1 - d \cos t$, $|\cos t| \leq 1$, and $d < 1$, we see that $dx/dt > 0$ and there are no such points.

(b) Find the tangent line to the curve at $t = \pi/2$.

Solution: The slope of a tangent line to the curve at $t = \pi/2$ is given by

\[
\frac{dy}{dt} \bigg|_{t=\pi/2} = \frac{d \sin (\pi/2)}{(1 - d \cos (\pi/2))} = d/(1 - 0) = d
\]

So the tangent line is $y = y(\pi/2) + d(x - x(\pi/2)) = 1 + d(x - \pi/2 + d)$.

(2) Find the tangential and normal components of the acceleration for a particle on the space curve $\vec{r}(t) = (2t, 1, t^2)$.

Solution: First calculate $\vec{r}'(t) = (2, 0, 2t)$ and $\vec{r}''(t) = (0, 0, 2)$. From class, or equation (7) on page 910, we know that $\vec{r}'' = v'\vec{T} + \kappa v^2 \vec{N}$ where $v = |\vec{r}'|$ is the (scalar) speed. In this case $v = |(2, 0, 2t)| = \sqrt{4 + 4t^2}$, so $v' = 4t/\sqrt{4 + 4t^2}$. The curvature is

\[
\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|(0,-4,0)|}{(4+4t^2)^{3/2}} = \frac{1}{2(t^2+1)^{3/2}}.
\]

(3) Find the curvature of $\vec{r}(t) = (t^2, t^3, 2t^3)$ at $t = 1$.

Solution: The first two derivatives are $\vec{r}$ are $\vec{r}' = (2t, 3t^2, 6t^2)$ and $\vec{r}'' = (2, 6t, 12t)$. At $t = 1$ these reduce to $(2, 3, 6)$ and $(2, 6, 12)$ respectively. Now we can compute the curvature:

\[
\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{6\sqrt{5}}{343}
\]

(The cross product at $t = 1$ is $\vec{r}' \times \vec{r}'' = (0, -12, 6)$.)

(4) Use the linearization of the function $f(x, y) = x + \ln xy$ at $(x, y) = (2, 1/2)$ to find an approximate value for $f(1.9, .4)$.

Solution: The linearization is

\[
L(x, y) = f(2, 1/2) + \frac{\partial f}{\partial x}(2, 1/2)(x - 2) + \frac{\partial f}{\partial y}(2, 1/2)(y - 1/2).
\]
\[2 + \ln 1 + (1 + \frac{1}{2})(x - 2) + (2)(y - 1/2) = -2 + 3x/2 + 2y\]

since \(\frac{\partial f}{\partial x} = 1 + 1/x\) and \(\frac{\partial f}{\partial y} = 1/y\).

So our approximation is \(L(1.9, 4) = 1.65 \approx f(1.9, 4) = 1.62556 \ldots\)

(5) Find \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\) from the implicit relation \(2x - z = \arctan(yz) + 1\).

Solution: For the \(x\) partial derivative, differentiate both sides to get

\[2 - \frac{\partial z}{\partial x} = \frac{y \frac{\partial z}{\partial x}}{1 + y^2 z^2}.\]

Solving for \(\frac{\partial z}{\partial x}\) we find \(\frac{\partial z}{\partial x} = \frac{2y^2 z^2 + 2}{y^2 z^2 + y + 1}\). Similarly, taking a \(y\) partial derivative gives us:

\[-\frac{\partial z}{\partial y} = \frac{z + \frac{\partial z}{\partial x}}{1 + y^2 z^2}\]

and \(\frac{\partial z}{\partial y} = -\frac{z}{y^2 z^2 + y + 1}\).

(6) Use the chain rule to compute \(\frac{\partial z}{\partial t}\) if \(z = \sin(xy) \sin(y)\) and \(x = 1/t, y = 2t\).

Solution:

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[= (y \cos(xy) \sin(y))(-\frac{1}{t^2}) + (x \cos(xy) \sin(y) + \sin(xy) \cos(y))(2)\]

\[= 2 \cos(2t) \sin(t)\]

after substituting for \(x\) and \(y\) and a lot of simplification.