Even Harmonious Graphs

by

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Abstract

A graph $G$ with $q$ edges is called \textit{graceful} if there is an injection $f$ from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. This notion as well as a number of other functions from a graph to a set of non-negative integers were studied as tools for decomposing the complete graph into isomorphic subgraphs. A graph $G$ with $q$ edges is said to be \textit{harmonious} if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices. Over the years many variations of these two concepts have been introduced and nearly 1000 articles have been published on them.

Recently two variants of harmonious labelings have been defined. A function $f$ is said to be an \textit{odd harmonious} labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of $G$ to the odd integers between 1 to $2q - 1$ is a bijection. A function $f$ is said to be an \textit{even harmonious} labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q$ and the induced function $f^*$ from the edges of $G$ to $\{0, 2, \ldots, 2(q-1)\}$ defined by $f^*(uv) = f(u) + f(v) \pmod{2q}$ is bijective.

Only a few papers have been written on even harmonious labelings. Therefore, I will focus on finding more graphs with even harmonious labelings. For example, I will investigate odd wheels, two stars joined by a path, the disjoint union two paths, the disjoint union of any number copies of a single edge, and the disjoint union of two wheels. Disjoint unions of graphs are of special interest because they permit more freedom in assigning the labels.
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1 Introduction

A vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ label depending on the vertex labels $f(x)$ and $f(y)$. Many graph labeling methods can be traced back to Rosa [RO] in 1967 and to Graham and Sloane [GS] in 1980. Harmonious graphs naturally arose in the study of error-correcting codes and channel assignment problems. Since then there have been numerous papers on harmonious labelings. However, the only paper published on even harmonious labelings is by Sarasija and Binthiya [SB] who introduced even harmonious labelings. An extensive survey of graph labeling methods is available online at [GA].

2 Preliminaries

Basic graph-theoretic terms are defined in the appendix.

Definition 1. A graph $G$ with $q$ edges is called graceful if there is an injection $f$ from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. See Figure 1.

![Figure 1: Graceful labelings of graphs](image)

Definition 2. A graph $G$ with $q$ edges is said to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When $G$ is a tree exactly one label may be used on two vertices. See Figures 2 and 3.

![Figure 2: A harmonious tree mod 5](image)
Definition 3. A function $f$ is said to be an odd harmonious labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of $G$ to the odd integers between 1 to $2q - 1$ is a bijection. See Figure 4.

Definition 4. A function $f$ is said to be an even harmonious labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q$ and the induced function $f^*$ from the edges of $G$ to $\{0, 2, \ldots, 2(q - 1)\}$ defined by $f^*(uv) = f(u) + f(v) \mod 2q$ is bijective. See Figure 5.

Definition 5. The join of graphs $G$ and $H$, $G + H$, is the graph obtained by joining every vertex of $G$ to every vertex of $H$.

Sarasija and Binthiya [SB] proved the following graphs are even harmonious: non-trivial paths; complete bipartite graphs; odd cycles; bistars $B_{m,n}$; $K_2 + K_n$; $P_n^2$; and the friendship graphs $F_{2n+1}$.

Definition 6. An even harmonious labeling of a graph with $q$ edges is called a properly even harmonious labeling if no vertex label is duplicated. See Figure 6.

A special kind of properly even harmonious labeling is strongly even harmonious labeling.

Definition 7. An even harmonious labeling of a graph with $q$ edges that satisfies the additional condition that for any two adjacent vertices with labels $u$ and $v$, $0 < u + v \leq 2q$ is called a strongly even harmonious labeling. See Figure 7.
So, in the case of a strongly even harmonious labeling of a graph with $q$ edges modular arithmetic is not done except for the case when the sum of two vertex labels is $2q$.

The following theorem will be used frequently. It follows from the fact that in a cyclic group of odd order the sum of the elements is 0 whereas in a non-trivial cyclic group of even order the sum of the elements is the unique element of order 2.

**Theorem 1.** For any even harmonious labeling of a graph with $q$ edges, the sum of the edge labels mod $2q$ is 0 when $q$ is odd and the sum is $q$ when $q$ is even.
Proof. When $q$ is odd the nonzero edge labels are $2, 4, \ldots, 2q - 4, 2q - 2$. Rearrange them as $2, 2q - 2, 4, 2q - 4, \ldots, q - 1, q + 1$. Observing that each successive pair sums to $0 \mod 2q$, we have the sum $0$.

When $q$ is even the nonzero edge labels are $2, 4, \ldots, q - 2, q, q + 2, \ldots, 2q - 4, 2q - 2$. Rearranging them as $2, 2q - 2, 4, 2q - 4, \ldots, q - 2, q + 2, q$. Notice the consecutive pairs up to the last term sum to $0 \mod 2q$. So, the sum is $q$ when $q$ is even. \hfill \Box

The following theorem shows that in an even harmonious labeling we can duplicate any label. It also shows that we can change the parity of the vertex labels from even to odd or vice versa.

**Theorem 2.** If $f$ is a (properly) even harmonious labeling for a graph $G$ with $q$ edges then for any unit $a$ in $Z_{2q}$ and any $b$ in $Z_{2q}$ the labeling $f^*(v) = af(v) + b$ is also a (properly) even harmonious labeling of $G$.

**Proof.** Let $v_1, v_2, \ldots, v_m$ be the vertices of $G$. Then the vertex labels of $f^*$ are $af(v_1) + b, af(v_2) + b, \ldots, af(v_m) + b$. Observe that $f^*(v_i) = af(v_i) + b = af(v_j) + b = f^*(v_j)$ if and only if $f(v_i) = f(v_j)$. To see that the edge labels induced by $f^*$ are distinct, observe that because $a$ is a unit $f^*(v_i) + f^*(v_j) = af(v_i) + b + af(v_j) + b = a(f(v_i) + f(v_j)) + 2b$ are distinct when the terms $f(v_i) + f(v_j)$ are distinct. \hfill \Box

Theorem 2 gives the following useful corollaries.

**Corollary 1.** In any even harmonious graph we may assume that the duplicate label is 0.

**Corollary 2.** In any connected even harmonious label we may assume the vertex labels are even.

**Definition 8.** The union of graphs $G_1$ and $G_2$, $G_1 \cup G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

A graph consisting of $k \geq 2$ disjoint copies of a graph $H$ is denoted by $kH$.

Note that for connected graphs any harmonious labeling of a graph with $q$ edges yields an even harmonious labeling by simply multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Thus we know that every connected harmonious graph is an even harmonious graph and every connected graph that is not a tree that has a harmonious labeling also has a properly even harmonious labeling. Consequently, we will focus on connected graphs that are not harmonious and disconnected graphs.

Also, note that for a connected graph with $q$ edges an even harmonious labeling that utilizes both 0 and $2q$ as vertex labels could have been obtained more simply finding a vertex labeling with integers from 0 to $q$ where each integer other than 0 or $q$ is used exactly once so that the induced edge labels obtained by adding the endpoints modulo $q$ are distinct then doubling the vertex labels. For an example see Figure 8.

### 3 Connected Graphs

**Theorem 3.** A tree (or forest) cannot have a properly even harmonious labeling.

**Proof.** Observe that a tree with $n$ vertices has $n - 1$ edges and is connected. In order for this to be a properly even harmonious labeling there cannot be a repeat of any vertex label. However, this is impossible since there are $n$ vertices with only $n - 1$ edges. Therefore, a tree (or forest) cannot have a properly even harmonious labeling since we will need to use a duplicate of some number. \hfill \Box
Definition 9. The wheel, $W_n$, is the graph obtained by joining every vertex of the cycle $C_n$ to exactly one isolated vertex called the center. The edges incident to the center are called spokes. See Figure 8.

Theorem 4. The wheel, $W_n$, is properly even harmonious when $n$ is odd.

Proof. Since $W_n$ has $2n$ edges the modulus is $4n$. Label the center vertex $v_0$ and label the consecutive cycle vertices $v_1, v_2, \ldots, v_n$. Label the vertex $v_0 = 2$. For $i \geq 1$, let $v_i = 4(i - 1)$. See Figure 8. Since these labels are strictly increasing and less than $4n$ they are distinct.

To verify that the edge labels are distinct observe that the cycle edges for $W_n$ have the form $4(i-1)+4i = 8i-4$. So if the edges $v_iv_{i+1}$ and $v_jv_{j+1}$ are equal we have $8i-4 = 8j-4 \mod 4n$. Thus $8(i-j) = k4n$ or $2(i-j) = kn$ for some $k$. Without loss of generality, we may assume $i > j$. Then $0 < i - j < n$ and $0 < kn = 2(i-j) < 2n$. Then $k = 1$ and $2(i-j) = n$. Since $n$ is odd we have a contradiction.

Since the spoke labels have the form $2 + v_i$ we have $2 + v_i \neq 2 + v_j$ whenever $v_i \neq v_j$. So the spoke labels are distinct.

Lastly, assume some cycle edge label say $8i - 4$ is the same as the spoke label $2 + v_j = 4j - 2$. Then we have $8i - 4 = 4j - 2 \mod 4n$, which is equal to $8i - 4j - 2 = 0 \mod 4n$. This equation simplifies to $-2 = 0 \mod 4n$, which is a contradiction. 

Definition 10. The helm $H_n$ is the graph obtained from a wheel by attaching a pendant edge at each vertex of the $n$-cycle.

Theorem 5. The helm, $H_n$, is properly even harmonious when $n$ is odd.

Proof. Since $H_n$ has $3n$ edges the modulus is $6n$. Denote the vertex of degree $n$ (the “center”) by $v_0$, the consecutive cycle vertices by $v_1, v_2, \ldots, v_n$, and the vertex of degree 1 adjacent to $v_i$ by $w_i$. First, label the center $v_0 = 0$. Next, label the cycle vertices $v_i = 6i - 2, i = 1, \ldots, n$.

Lastly, label the outer most vertices with $w_i = 6i - 6, i = 1, \ldots, n$.

Note that $v_0 = 2$; the $v_i$’s are $4, 10, 16, \ldots, 6n - 2$; and the $w_i$’s are $0, 6, 12, \ldots, 6n - 6$. So all vertex labels are distinct.

Next note that the spoke $v_0v_i$ has the label $6i$; the cycle edge $v_iv_{i+1}$ has the label $12i - 10$; and the pendant edge $v_iiw_i$ has the label $12i - 8$. Before reducing modulo $6n$, this yields the following edge labels: spokes: $6, 12, 18, \ldots, 6i, \ldots, 6n$; cycle edges: $14, 26, 38, \ldots, 12i - 10, \ldots, 12n - 10$; pendant edges: $4, 16, 28, \ldots, 12i - 8, \ldots, 12n - 8$. By observation, these are distinct mod $6n$ (each type has a different remainder modulo 6). See Figure 9.
Graham and Sloane [GS] proved that $C_n$ is harmonious if and only if $n$ is odd. So we will consider harmonious labelings for the case when $n$ is even. Although we are not able to prove all cycles of the form $C_{4n}$ are even harmonious we can prove the following.

**Theorem 6.** The graph, $C_{2n}$, is not even harmonious when $n$ is odd.

**Proof.** Suppose $n$ is odd. Rosa [RO] proved that when $n$ is odd $C_{2n}$ does not have a properly even harmonious labeling. By Corollary 1 of Theorem 2, we may assume that two of the labels are 0 and 0. Now if we have an even harmonious labeling we know the edge labels are 0, 2, 4, $\ldots$, $4n - 2$. When we add these mod 4 we get $2n$ by Theorem 1. But we know that the sum of the edge labels is just the sum of the vertices where each vertex appears exactly two times. Say that $x$ is the missing nonzero label in $C_{2n}$. Now look at the sum of all the edges of the labels we use. It must be of the form $2(0 + 0 + 2 + \cdots + 4n - 2) - 2x$ since every entry in the sum appears exactly twice except $x$. But notice that $2(0 + 0 + 2 + \cdots + 2n - 2 + 2n + 2 + \cdots + 4n - 2) - 2x = 0 \mod 4n$. Moreover, since each pair $2, 4n - 2; 4, 4n - 4; \ldots, 2n - 2, 2n + 2$ sum to 0 mod 4n we have $2n - 2x = 0 \mod 4n$. This reduces to $n - x = 0 \mod 2n$ but $n$ is odd and $x$ is even, which is a contradiction. 

In contrast to Theorem 6, the labeling 0, 0, 2, 4 is an even harmonious labeling for $C_4$. Moreover, a computer search shows that $C_8, C_{12}, C_{16}, C_{20}$ and $C_{24}$ are even harmonious. Here are the even harmonious labels for these graphs.

$C_8$: 6, 2, 14, 0, 10, 8, 12, 0 (See Figure 10)
$C_{12}$: 20, 0, 8, 16, 22, 18, 10, 12, 0, 2, 4, 14
$C_{16}$: 20, 2, 16, 4, 22, 12, 24, 6, 0, 14, 18, 26, 30, 10, 0, 28
$C_{20}$: 36, 4, 28, 14, 16, 32, 12, 26, 34, 2, 8, 20, 6, 0, 22, 30, 24, 0, 18, 38
$C_{24}$: 36, 42, 40, 2, 22, 44, 30, 28, 20, 0, 4, 24, 14, 0, 32, 8, 38, 46, 10, 6, 16, 34, 26, 18

We conjecture that $C_{4n}$ is harmonious for all $n$. Notice that for each of these six cycles the missing label is 2$n$. Although we can not prove $C_{4n}$ is even harmonious, we can say the following.

**Theorem 7.** For any even harmonious labeling of $C_{4n}$ the label not used is 2$n$ or 6$n$. 

Figure 9: Properly even harmonious labeling of the helm $H_5$ mod 30
Proof. The modulus is $8n$. Recall that $C_{4n}$ does not have a proper even harmonious labeling. So we may assume that two of the labels are 0 and 0. Now if we have an even harmonious labeling we know the edges are $0, 2, 4, \ldots, 8n - 2$. When we add these mod $8n$ we get $4n$ (see Theorem 1). But we know that the edges are just the sum of the vertices where each vertex appears exactly two times. Say that $x$ is the missing nonzero label in $C_{4n}$. Now look at the sum of all the edges of the labels we used. It must be of the form $2(0 + 0 + 2 + 4 + \cdots + 8n - 2) - 2x$ since every entry in the sum appears exactly twice except $x$. But notice that $2(0 + 0 + 2 + 4 + \cdots + 8n - 2) = 2(4n) - 2x = -2x \mod 8n$. So, we have that $-2x = 4n \mod 8n$. Solving for $x$ we get $x = 2n$ or $x = 6n$.

Another example of a connected graph that is not harmonious but is even harmonious is shown in Figure 11. The fact that $C_6$ with a cord joining two vertices at distance 2 apart is not harmonious was proved in [XU].

We next investigate whether $K_n$ is even harmonious. The cases $n = 2$ and 3 are trivial. The case $k = 4$ is given in Figure 12. The result of Graham and Sloane that $K_m$ is harmonious if and only if for $m \leq 4$ (see the next to last paragraph of Section 2 in [GS]) also settles the question of which complete graphs are even harmonious.

**Theorem 8.** $K_n$ is even harmonious if and only if $n \leq 4$.

**Proof.** By the result of Granham and Sloane, any even harmonious labeling of $K_n$ would have $n > 4$ and have a duplicate vertex label $x$. But then for any vertex label $y$ other than $x$ the edge label $x + y$ is used twice.

**Theorem 9.** A graph obtained by identifying exactly one vertex from any finite number of complete graphs (the one-point union) where each has order at least 3 (they need not have the same order) is
even harmonious if and only if it is harmonious.

Proof. Since the graph is connected we need only consider the case where the graph is not harmonious but is even harmonious. By Corollaries 1 and 2 of Theorem 2, we may assume labels are even and we may assume the duplicate label is 0. If the two vertices labeled 0 are on the same complete graph then for any nonzero label \( x \) on that graph the edge \( x \) appears twice. If the two 0’s are on different cycles and the vertex where the cycles are joined is labeled \( x \), then \( x \) appears twice on edge label.

Since Graham and Sloane [GS] have proved that the one-point union two copies of \( K_n \) for \( n \geq 3 \) and odd is not harmonious [GS] we have the following corollary.

**Corollary 3.** The one-point union of two copies of \( K_n \), where \( n \geq 3 \) and \( n \) is odd is not even harmonious.

**Theorem 10.** The graph obtained by identifying one vertex from each of the two copies of \( K_n \), where \( n \geq 3 \) and \( n \) is odd is not even harmonious.

Proof. By Corollaries 1 and 2, we may assume labels are even and we may assume the duplicate label is 0. A proper even harmonious labeling is impossible since it has been proved that the graph obtained by identifying one vertex of the two copies of \( K_n \) for \( n \geq 3 \) and odd is not harmonious [GS]. If the two vertices labeled 0 are on the same complete graph then for any nonzero label \( x \) on that graph the edge \( x \) appears twice. If the two 0’s are on different cycles and the vertex where the cycles are joined is labeled \( x \), then \( x \) appears twice on edge label.

4 Disconnected Graphs

**Observation 1.** Note that for an even harmonious labeling of a connected graph all the vertex labels must have the same parity but for disconnected graphs different components can have different parities. This is especially important because we can then use both even and odd labels. With connected graphs all evens or all odds had to be used. For disconnected graphs we can use even labels only on the components, odd labels only on the components, or odd labels on some components and even labels on other components.

**Theorem 11.** The matching \( nP_2 \) is properly even harmonious if and only if \( n \) is even.

Proof. First suppose that \( n \) is even. Drawing the graph as shown in Figure 13, label the vertices starting in the top left corner to the bottom left corner with \( v_1, v_2, ..., v_n \) then start at the top right corner to the bottom right corner with \( v_{n+1}, v_{n+2}, ..., v_{2n} \). To label the vertices, let \( v_i = (i-1) \mod 2n. \)
By observation, the edge labels are \(n, n+2, \ldots, 2n-2; 0, 2, \ldots, n-2\). See Figures 13 and 14.

We now suppose \(n\) is odd, and show \(nP_2\) is not properly even harmonious. The \(n-1\) even integers can be used on at most \((n-1)/2\) edges. Likewise, the odds can be used on at most \((n-1)/2\) edges. This leaves one edge that uses the remaining two labels, one of which is even and odd, but then one edge label is odd.

\[\begin{array}{c}
0 & \rightarrow & n \\
1 & \rightarrow & n+1 \\
2 & \rightarrow & n+2 \\
\vdots & \vdots & \vdots \\
n/2 & \rightarrow & n+n/2 \\
n/2+1 & \rightarrow & n+n/2+1 \\
\vdots & \vdots & \vdots \\
n-1 & \rightarrow & 2n-1 \\
\end{array}\]

Figure 13: Properly even harmonious labeling of \(nP_2\) mod \(2n\) where \(n\) is even

**Theorem 12.** The graph \(nP_2\) is even harmonious if \(n\) is odd.

**Proof.** Drawing the graph as shown in Figure 13, label the vertices starting in the top left corner to the bottom left corner with 0, 1, \ldots, \(n-1\) then start at the top right corner to the bottom right corner with \(n-1, n, \ldots, 2n-2\). Observe that all of the vertex labels are distinct except for the one repeat. The edge labels are \(n-1, n+1, \ldots, 2n-2; 0, 2, \ldots, n-3\). Obviously the edge labels are distinct. See Figure 15.

**Theorem 13.** The graph \(S_m \cup P_n\) is strongly even harmonious if \(n \geq 2\).

**Proof.** The modulus is \(2m + 2n - 2\).

Step 1: Label the center of the star 0. Label the vertices of the star 2, 4, 6, \ldots.

Step 2: Starting with the first vertex on \(P_n\) label the vertices 1, 3, 5, \ldots skipping a vertex each time. Starting with the second vertex label the vertices \(2m + 1, 2m + 3, \ldots\) skipping a vertex each time.

The only modular arithmetic used is for the last edge of \(P_n\), therefore all vertex labels and edge labels are distinct. See Figure 16.

**Theorem 14.** The graph \(S_{m_1} \cup S_{m_2} \cup P_n\) is properly even harmonious if \(4 \leq n < 2m_1 + 2m_2 + 1\).
Proof. The modulus is $2m_1 + 2m_2 + 2n - 2$. We may assume $m_1 \geq m_2$.

Step 1: Label the center of $S_{m_1}$ with 0. Then label the outside vertices with $0, 2, 4, \ldots, 2m_1$

Step 2: Label the center of $S_{m_2}$ with $2m_1 + 2m_2 + 2n - 4$. Then label the outside vertices with $2m_1 + 4, 2m_1 + 6, 2m_1 + 8, \ldots, 2m_1 + 2m_2 + 2$.

Step 3: Starting with the first vertex of $P_n$ label the vertices 1, 3, 5, \ldots skipping a vertex each time. Starting with the second vertex label the vertices $2m_1 + 2m_2 + 1, 2m_1 + 2m_2 + 3, 2m_1 + 2m_2 + 5, \ldots$ skipping a vertex each time. See Figure 17. \hfill \Box

Theorem 15. The graph $C_n \cup P_3$ where $n \geq 3$ and $n$ is odd is properly even harmonious.

Proof. The modulus is $2n + 4$.

Case i: $n = 3$. See Figure 18 for $C_3 \cup P_3$.

Case ii: $n > 3$.

Step 1: Label the consecutive labels of $C_n$ with $v_i = 2i, i = 0, 1, \ldots, n - 1$ when $n > 3$. Clearly, the
vertex labels are distinct. The edge labels for $C_n$ are $2, 6, 10, \ldots, 2n + 2, 0, \ldots, 2n - 10, 2n - 2$.

Step 2: To label $P_3$, let $x$ and $y$ be the edges labels not appearing in $C_n$. Label the middle vertex of $P_3$ with a 1. Label one of the remaining vertices of $P_3$ with $x - 1$ and the other vertex $y - 1$. See Figure 19.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure18.png}
\caption{Properly even harmonious labeling of $C_3 \cup P_3$ mod 10}
\end{figure}

**Theorem 16.** The graph $C_4 \cup P_n$ is properly even harmonious for all $n \geq 2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure16.png}
\caption{Strongly even harmonious labeling of $S_5 \cup P_7$ mod 22}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure17.png}
\caption{Properly even harmonious labeling of $S_8 \cup S_4 \cup P_7$ mod 36}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure18.png}
\caption{Properly even harmonious labeling of $C_3 \cup P_3$ mod 10}
\end{figure}
**Proof.** The modulus is $2n + 6$.

Case i: $n = 2, 3, 4$.

Step 1: Label the vertices of $C_4$ with $0, 2, 2n + 4$, and $6$ in order. Because the mod is $2n + 6$ this gives us the edges $2, 0, 6, 4$ respectively.

Step 2: Starting with the left end point of $P_n$ label every other vertex with $1, 3, 5, \ldots$ (up to $\lceil \frac{n}{2} \rceil$ terms). Then start with the second vertex from the left and label every other vertex with $7, 9, 11, \ldots$ (up to $\lfloor \frac{n}{2} \rfloor$ terms).

We illustrate the labeling for $n = 4$ in Figure 20.

Case ii: $n \geq 5$.

Step 1: Label the vertices of $C_4$ with $1, 2n - 5, 3$, and $2n - 1$ in order.

Step 2: Starting with the left end point of $P_n$ label every other vertex with $6, 8, 10, \ldots$ Then start with the second vertex from the left and label every other vertex with $2n - 2, 2n, 2n + 2, \ldots$ The edges labels on $P_n$ are $2n + 4, 0, 2, \ldots, 4n - 2, 4n$ and the edge labels on $C_4$ are $4n + 2, 4n + 4, 4n + 6, 4n + 8$.

It is clear to see now that the edge labels are distinct. See Figure 21. 

\[ \square \]
The join of graphs $G$ and $H$, $G + H$, is obtained by joining every vertex of $G$ with every vertex of $H$ with an edge. The fan, $F_n$, is the graph $P_n + K_1$.

**Theorem 17.** The graph $C_4 \cup F_n$ is properly even harmonious for $n > 1$.

**Proof.** The modulus is $4n + 6$.

Case i: $n$ is odd.

Step 1: Start at the first vertex on $F_n$ with 1 than alternate by increments of 2. Start at the second vertex with $n + 2$ and alternate with increments of 2. Label the remaining vertex label with $3n$.

Step 2: Label $C_4$ with $4n + 4, n - 3, 2, n - 1$. See Figure 22.

Case ii: $n = 2$ and $n = 4$ are shown in Figures 23 and 24.

Case iii: $n$ is even, $n > 4$.

Step 1: Start at the first vertex on $F_n$ with 1 than alternate by increments of 2. Start at the second vertex with $n + 1$ and alternate with increments of 2. Label the remaining vertex label with $3n - 1$.

Step 2: Label $C_4$ with $0, n - 4, 4n + 4, n$ in order. See Figure 25.

\[\square\]

![Figure 22: Properly even harmonious labeling of $C_4 \cup F_7$ mod 34](image)

![Figure 23: Properly even harmonious labeling of $C_4 \cup F_2$ mod 14](image)

**Theorem 18.** The graph $K_{m,2} \cup P_n$ is properly even harmonious for $1 < n < 4m + 3$. 

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Figure 24: Properly even harmonious labeling of $C_4 \cup F_4 \mod 22$

Figure 25: Properly even harmonious labeling of $C_4 \cup F_6 \mod 30$

Proof. The modulus is $4m + 2n - 2$.
Case i: $2 \leq n < 2m + 1$.
Step 1: Denote the partite set with 2 vertices by $A$ and the partite set with $m$ vertices by $B$. Label one vertex of $A$ with 0 and the other vertex of $A$ with $2m + 2$. Then label the vertices of $B$ with 2, 4, 6, . . . , $2m$.
Step 2: Label the first vertex of $P_n$ with $2m + 1$. Starting with the second vertex of $P_n$ label the vertices 1, 3, 5, . . . , $n - 1$ or $n - 2$ depending on whether $n$ is even or odd skipping a vertex each time. Starting with the third vertex of $P_n$ label the vertices $4m + 3, 4m + 5, 4m + 7, \ldots$
Our missing edge label on $K_{m,2}$ is $2m + 2$. On $P_n$ the first and second vertices are $2m + 1$ and 1 respectively. To avoid using the vertex label $2m + 1$ a second time, the vertex labels of $P_n$ in the even numbered positions must stop before we reach the value $2m + 1$. That is, $n < 2m + 1$. See Figure 26.

Case ii: $2m + 2 < n < 4m + 3$.
Step 1: Label one vertex of $A$ with 0 and the other vertex of $A$ with 2. Then label the vertices of $B$ with $2n - 4m - 2, 2n - 4m + 2, \ldots , 2n - 6$. The smallest edge label of $K_{m,2}$ is $2n - 4m - 2$. The largest edge label of $K_{m,2}$ is $2n - 4$.
Step 2: Label the first vertex of $P_n$ with $2n - 3$ then skipping a vertex each time increase by increments
of 2. Starting with the second vertex of $P_n$ label the vertices $1, 3, 5, \ldots, n - 1$ or $n - 2$ (depending on whether $n$ is even or odd) skipping a vertex each time the first edge label of $P_n$ is $2n - 2$. The last edge label of $P_n$ is $2n - 4m - 4$. On $P_n$ the first and second vertices are $2m - 3$ and $1$ respectively. To avoid using the vertex label $1 = 4m + 2n - 1 \mod (4m + 2n - 2)$ a second time, the vertex labels of $P_n$ in the even numbered positions must stop before we reach the value $4m + 2n - 1$. Observing that when $n$ is even the last vertex label in the odd numbered positions is $2n - 3 + n - 2 = 3n - 5$, we must have $3n - 5 < 4m + 2n - 1$ or $n < 4m + 4$. Likewise, when $n$ is odd, the last vertex label in the odd numbered positions is $2n - 3 + n - 1 = 3n - 4$ so we must have $3n - 4 < 4m + 2n - 1$ or $n < 4m + 3$. To handle both cases simultaneously it suffices to take $n < 4m + 3$. See Figure 27.

Case iii: $n = 2m + 1$ or $n = 2m + 2$.

Step 1: Label one vertex of $A$ with 0 and the other vertex of $A$ with $4m + 2n - 4$. Then label the vertices of $B$ with $2, 6, \ldots, 4m - 2$.

Step 2: Starting with the first vertex of $P_n$, label the vertices $1, 3, 5, \ldots, n - 1$. Label the second vertex off $P_n$ with $4m - 1$ then skipping a vertex each time increase by increments of two. See Figure 28.

\[ \begin{align*}
&\text{Figure 26: Properly even harmonious labeling of $K_{4,2} \cup P_8 \mod 30$} \\
&\text{Figure 27: Properly even harmonious labeling of $K_{4,2} \cup P_{13} \mod 40$}
\end{align*} \]

\textbf{Theorem 19.} The graph $P_m \cup P_n$ is properly even harmonious for all $m \geq 2$ and $n \geq 2$. 

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Proof. We may assume that $m \geq n$. The modulus is $2m + 2n - 4$.

Step 1: Label every other vertex of $P_m$ with $1, 3, 5, \ldots$ wrapping around. By construction the edge labels are distinct. For the case that $n = 2$ there will be a single edge label missing, call it $x$. If $x \neq 0$, label $P_2$ with 0 and $x$. If $x = 0$ and $m > 2$, label $P_2$ with $2m - 2$ and 2. If $m = 2$, label $P_2$ with 0 and 2.

Step 2: For $n > 2$, let $r = 2\lceil \frac{m}{4} \rceil$.

Case i: $m$ is odd.
Note that the last edge label of $P_m$ is $3m - 1 \mod (2m + 2n - 4)$. Label the first vertex of $P_n$ with $r$. Label the second vertex of $P_n$ with $3m + 1 - r$. This results in the first edge label of $P_n$ of $3m + 1 \mod (2m + 2n - 4)$. Then label every other vertex in an odd numbered position by increments of 2 and every other vertex in an even numbered position by increments of 2. See Figure 29.

To verify that this is a properly even harmonious labeling observe that for a duplicate vertex label of $P_n$ to occur it is necessary that $3m + 1 - r + n - 3 = r$ which converts

$$3m + 1 + n - 3 = 2r = \begin{cases} m + 3 & \text{when } m \mod 4 = 1 \\ m + 1 & \text{when } m \mod 4 = 3. \end{cases}$$

When $2r = m + 3$ we have $3m + 1 + n - 3 = m + 3$ or $2m + n - 5 = 0 \mod (2m + 2n - 4)$. But for $n \geq 3$, we have $2m + n - 5 < 2m + n - 1 \leq 2m + 2n - 4$, which is a contradiction since $2m + 2n - 5 > 1$.
When $2r = m + 1$ we have $3m + 1 + n - 3 = m + 1$ of $2m + n - 3 = 0 \mod (2m + 2n - 4)$. But for $n \geq 1$, we have $2m + n - 3 < 2m + n - 1 \leq 2m + 2n - 4$, which is a contradiction.

Figure 29: Properly even harmonious labeling of $P_7 \cup P_7 \mod 24$ with $r = 4$
Case ii: $m$ is even.

Note that the last edge of $P_m$ is $3m - 2 \mod (2m + 2n - 4)$. Label the first vertex of $P_n$ with $r$. Label the second vertex of $P_n$ with $3m - r \mod (2m + 2n - 4)$. Label the remaining vertices of $P_n$ as in the case that $m$ is odd. For both cases of $m$ the edge labels are distinct since there is no wrap around modulo $2m + 2n - 4$. By construction the vertex labels on $P_m$ are distinct. To verify that the vertex labels for $P_n$ are distinct we consider two cases. See Figure 30.

To verify that this is a properly even harmonious labeling observe that for a duplicate vertex for $P_n$ to occur it is necessary that $3m - r + n - 2 = r$ which converts to

$$3m + n - 2 = 2r = \begin{cases} m + 2 & \text{when } m \mod 4 = 1 \\ m + 1 & \text{when } m \mod 4 = 3. \end{cases}$$

When $2r = m + 3$ we have $3m + 1 + n - 3 = m + 3$ or $2m + n - 5 = 0 \mod (2m + 2n - 4)$. But for $n \geq 3$, we have $2m + n - 5 < 2m + n - 1 \leq 2m + 2n - 4$, which is a contradiction. When $2r = m + 1$ we have $3m + 1 + n - 3 = m + 1$ of $2m + n - 3 = 0 \mod (2m + 2n - 4)$. But for $n \geq 1$, we have $2m + n - 3 < 2m + n - 1 \leq 2m + 2n - 4$, which is a contradiction. \( \square \)

![Figure 30: Properly even harmonious labeling of $P_8 \cup P_8$ mod 22 with $r = 4$](image)

**Theorem 20.** $S_{n_1} \cup S_{n_2}$ is strongly even harmonious when $n_1 > n_2$ and $n_1 > 3$.

**Proof.** The modulus is $\mod 2(n_1 + n_2)$.

Step 1: Label the center vertex of $S_{n_1}$ with 0.

Step 2: Label all of the rest of the vertices with even numbers starting with 2 and increase with increments of two in order.

Step 3: Label the center of the remaining star with 1.

Step 4: Label all the rest of the vertices with odd numbers starting with $2n_1 + 1$ and increase with increments of two. \( \square \)

**Theorem 21.** $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious for $n_1 \geq n_2, \geq \ldots \geq n_t$ and $t < \frac{n_t}{2} + 2$.

**Proof.** Let $n = n_1 + n_2 + \cdots + n_t$. The modulus is $\mod 2n$.

Step 1: Label the center vertex of $S_{n_1}$ with 0 and the centers of $S_{n_1}, \ldots, S_{n_3}$ and $S_{n_2}$ with 1, 3, $\ldots$, $2t-3$, respectively.

Step 2: Label the end vertices of $S_{n_1}$ with 2, 4, $\ldots$, $2n_1$.

To label the remaining vertices, begin with $S_{n_1}$ and recursively label the other stars from right to left as described below. See Figure 31.

Step 3: Label the end vertices of $S_{n_1}$ with $2n - 1, 2n - 3, \ldots, 2n - 1 - 2(n_t - 1)$.

Step 4: Label one end vertex of $S_{n_{t-1}}$ with 4 less than the smallest end vertex label of $S_{n_t}$. (That is, $2n - 1 - 2(n_t - 1) - 4$). Label the remaining end vertices of $S_{n_{t-1}}$ successively with increments of -2. The smallest end vertex label of $S_{n_{t-1}}$ will be $2n - 1 - 2(n_t - 1) - 4 - 2(n_{t-1} - 1)$.

Step 4: Continue to move to each star to the left by labeling one end vertex with 4 less than the smallest end vertex label of the previous star and labeling the remaining end vertices of that star using

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increments of $-2$. The smallest label of the end vertices of $S_{n_2}$ will be $2n - 1 - 2(n_2 + n_3 + \cdots + n_t) + 2(t - 1) - 4(t - 2) = 2n_1 - 2t + 5$. Since the label of the center of $S_2$ is $2t - 3$ we may avoid a duplicating a vertex label by taking $2n_1 - 2t + 5 > 2t - 3$, which simplifies to $t < \frac{n_1}{2} + 2$.

$$\text{Figure 31: Strongly even harmonious labeling of } S_6 \cup S_5 \cup S_4 \cup S_4 \mod 38 \text{ n = 19}$$

Although the argument in the proof of Theorem 20 does not handle the case when $t \geq \frac{n_1}{2} + 2$, other methods might work. Figure 32 shows one such example.

**Conjecture.** $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious if at least one star has more than 2 edges.

**Theorem 22.** For $1 < n < 21$, $W_4 \cup P_n$ is even harmonious.

*Proof.* The modulus is $2n + 14$.

Case i: $n = 2$

$W_4 \cup P_2$ the vertex labels are 3 and 17 on $P_2$. On $W_4$ label the cycle 0, 6, 10, 8 and label the center 4.

Case ii: $n \geq 3$

Step 1: On $W_4$ label the cycle 0, 6, 10, 8 and label the center 4.

Step 2: On $P_4$ start with the first vertex and label 1, 3, 5, ... skipping a vertex each time. Now starting at the second vertex and skipping a vertex each time use 19, 21, 23, ... See Figure 33.

**Theorem 23.** The graph $K_4 \cup P_n$ is properly even harmonious when $2 \leq n \leq 12$.

*Proof.* The modulus is $2n + 10$.

Step 1: Label the outside vertices of $K_4$ in this order 0, 2, 4, 8.

Step 2: Starting with the first vertex of $P_n$ then skipping a vertex each time use the labels 1, 3, 5, ... Starting at the second vertex and skipping a vertex each time use 13, 15, 17, ...
The edge labels on the outside of $K_4$ are 2, 6, 8, 12 and on the inside the labels are 4 and 10. The edges on $P_n$ are $14, 16, 18, \ldots, 2n + 10$. See Figure 34.

We remark that the method used in Theorem 22 provides even harmonious labeling for $K_4 \cup P_n$ where the vertex label 13 is used twice.

**Theorem 24.** The graph $C_3 \cup P_n^2$ is properly even harmonious when $n \geq 2$. 

Proof. The modulus is $4n$.

Step 1: Label the vertices on $C_3$ with $0, 2, 4$ in order.
Step 2: Starting with the left endpoint of $P^2_n$ label the vertices with $3, 5, 7, \ldots$ in order. See Figure 35.

Figure 35: Properly even harmonious labeling of $C_3 \cup P^2_n \mod 20$

**Theorem 25.** The graph $C_4 \cup P^2_n$ is properly even harmonious when $n \geq 2$.

Proof. The modulus is $4n + 2$.

Step 1: Label the vertices on $C_4$ with $0, 2, 4n, 6$ in order.
Step 2: Starting with the left endpoint of $P^2_n$ label the vertices with $3, 5, 7, \ldots$ in this order. See Figure 36.

Figure 36: Properly even harmonious labeling of $C_4 \cup P^2_n \mod 22$

**Theorem 26.** The graph $P^2_m \cup P_n$ is even harmonious when $2 \leq n < 4m - 5$ and $m \geq 2$.

Proof. The modulus is $2(m + n - 1)$.

Step 1: Label $P^2_m$ with $0, 2, 4, \ldots, 2m - 2$ in order.
Step 2: Label $P_n$ starting with the first vertex of $P_n$ and label $1, 3, 5, \ldots$ skipping a vertex each time. Start at the second vertex and skipping a vertex each time use $4m - 5, 4m - 3, 4m - 1, 4m + 1, \ldots$ in order. Then skipping a vertex each time use the labels $1, 3, 5, \ldots$. To avoid using the vertex label
4m − 5 a second time, the vertex labels of \( P_n \) in the odd numbered position must stop before we reach the value 4m − 5. That is, \( n < 4m − 5 \). See Figure 37.

![Figure 37: Even harmonious labeling of \( P_5^2 \cup P_4 \mod 20 \)](image)

5 Disjoint Unions of Cycles

In this section we provide a few results on the disjoint union of cycles. Recall that a connected graph has a properly even harmonious labeling if and only if it has a harmonious label. Examples 1 and 2 are disconnected graphs that are not harmonious but have even harmonious labelings. Example 2 shows a graph that is not harmonious but is properly even harmonious.

**Example 1.** The graph \( C_3 \cup C_4 \) is not harmonious but is properly even harmonious.

**Proof.** We know that \( C_3 \cup C_4 \) is not harmonious from [Seoud]. Figure 38 shows that it is properly even harmonious.

![Figure 38: Properly even harmonious labeling of \( C_3 \cup C_4 \mod 14 \)](image)

**Example 2.** The graph \( C_4 \cup C_4 \) is not harmonious but is properly even harmonious.

**Proof.** We know that \( C_4 \cup C_4 \) is not harmonious from [Seoud]. Figure 39 shows that it is properly even harmonious.

**Theorem 27.** \( nC_3 \) is a properly even harmonious graph for all \( n \).

**Proof.** The modulus is 6n.

Step 1: Label the first \( C_3 \) with 1, 3, 6n − 1. Starting at the top and moving clockwise.

Step 2: Label the next \( C_3 \) with 2, 4, 6.

Step 3: Label the next \( C_3 \) with 5, 7, 9.

Step 4: Continue labeling each \( C_3 \) with the smallest remaining positive integers and in order alternating from even to odd for each \( C_3 \). For \( n > 1 \), the \( n \)th triangle is labeled 3n − 4, 3n − 2, 3n. The edge labels are 6n − 6, 6n − 4, 6n − 2. See Figure 40.
Theorem 28. The graph $kC_4$ is even harmonious for $1 \leq k \leq 6$.

Proof. The modulus is $8k$.
In Figure 41, the first $k$ squares shows the labeling for $kC_4$. \qed
6 Further Research

The following families are candidates for investigation.
1. $K_4 \cup P_m \cup P_n$
2. $P_m^2 \cup K_5$
3. $kC_n$ $n$ is odd.
4. $C_m \cup P_n$ when $m$ is odd and $m \geq n$.
5. $C_m \cup S_n$
6. $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$
7. $P_s \cup P_t \cup P_u$
References


Appendices

Definition 11. A cycle on the vertices \( v_1, v_2, \ldots, v_n \) is the graph denoted by \( P_n \) with the edge set \( \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\} \). The cycle, \( C_n \), is an \( n \) cycle.

![Figure 42: An even harmonious graph of \( C_4 \)](image)

Definition 12. A path on the vertices \( v_1, v_2, \ldots, v_n \) is the graph denoted by \( P_n \) with the edge set \( \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\} \). The distinct vertices \( v_1 \) and \( v_n \) are called the end vertices. The path \( P_n \) is said to have length \( n - 1 \).

Definition 13. A complete \( k \)-partite graph \( G \) is a \( k \)-partite graph with partite sets \( V_1, V_2, \ldots, V_k \) having the added property that if \( u \in V_i \) and \( v \in V_j, i \neq j \), then \( uv \in E(G) \). If \( |V_i| = n_i \), then this graph is denoted by \( K_{n_1, n_2, \ldots, n_k} \).

Definition 14. A complete bipartite graph with partite sets \( V_1 \) and \( V_2 \), where \( |V_1| = m \) and \( |V_2| = n \), is denoted by \( K_{m,n} \).

Definition 15. The graph \( K_{1,n} \) is called a star and is often denoted by \( S_n \). The vertex joining the \( n \) vertices is called the center.

Definition 16. A graph is a complete multipartite graph if it is a complete \( k \)-partite graph for some \( k \geq 2 \).

Definition 17. An acyclic graph has no cycles.

Definition 18. A tree is an acyclic connected graph.

Definition 19. A forest is an acyclic graph.

Definition 20. A vertex \( u \) is said to be connected to a vertex \( v \) in a graph \( G \) if there exists a \( u - v \) path in \( G \). A graph \( G \) is connected if every two if its vertices are connected. A graph that is not connected is disconnected.

Definition 21. A component of a graph \( G \) is a connected subgraph of \( G \) not properly contained in any other connected subgraph of \( G \).