Econ 3023 Microeconomic Analysis

Chapter 19:
Profit Maximization Problem

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Introduction

Short-Run Profit Maximization Problem

Comparative Statics

Long-Run Profit Maximization Problem

Factor Demand

Returns to Scale & Profit Maximization Problem

Summary
Question 1.1 (Agenda for Today)

- What is Jack's hiring and investment plan?
  - In the short run and long run. How do they differ?
- How does Jack respond to environmental change?
- What is his labor demand?
- What would be the limitation of profit maximization problem?

Assumption 1.2 (Firm's Objective)

- The firm's objective is to maximize their_____.
- Bake as many as you can.

Firm's Objective

- Hiring too many chefs will reduce the productivity eventually.
- How does Jack find the right production plan?
  - Dumb Jack: trials and errors.
  - Smart Jack: _____
Definitions

- \( w = (w_C, w_K) \) denotes the **unit price of inputs**.
- \( w = (10, 1) \) means hourly wage is $10 and rental rate is $1.

**Definition 1.3 (Total Cost)**

Associated with the input bundle \((x_C, x_K)\) is

\[
TC(x_C, x_K) = \ldots
\]

- If Jack hires 20 chefs and purchases 10 stand mixers under \(w = (1, 1)\),

\[
TC(x_C = 20, x_K = 5) = 1 \cdot 20 + 1 \cdot 5 = 25.
\]

All the costs are measured in terms of **opportunity cost**.

Jack’s financial capital (interest rate = 10%):

<table>
<thead>
<tr>
<th>A unit of kitchen</th>
<th>Self-financed</th>
<th>Borrowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out-of-pocket</td>
<td>10K</td>
<td>0K</td>
</tr>
<tr>
<td>Loan</td>
<td>0K</td>
<td>10K</td>
</tr>
<tr>
<td>Accounting Cost</td>
<td>___</td>
<td>___</td>
</tr>
<tr>
<td>Opportunity Cost</td>
<td>___</td>
<td>___</td>
</tr>
</tbody>
</table>

- \( w_K \) is not ___ but ___.

**Definition 1.4 (Total Revenue)**

**Total revenue** from \(y\) is

\[
TR(y) = \ldots \quad \text{or} \quad TR(x_C, x_K) = pf(x_C, x_K).
\]

- If Jack produces 10 cheesecakes and price is $4, his total revenue is $\ldots$. 
Definitions

Definition 1.5 (Profit)
The economic profit generated by the production plan \((x_C, x_K, y)\) is

\[ \pi(x_C, x_K) = \]

- If Jack raised $40 from cheesecakes sales and paid $25 for his employees and kitchen investment, his profit is ___.

Example 1.6 (Profit Structure)
Jack produces cheesecakes \(y\) according to \(y = \frac{1}{2}x_C \cdot x_K\). Factor price is \(w = (w_C, w_K) = (1, 1)\) and cheesecake sells for $4 apiece. If he hires 20 chefs and rents out 5 stand mixers, his
- labor cost is ___.
- capital cost is ___.
- total cost is ___.
- sales volume is ___.
- total revenue is ___.
- profit is ___.

Price-Taker Assumption

Definition 1.7 (Competitive Market)
Jack is in a competitive market if he is a

- Competitive labor market: \(w_C\) is given.
- Competitive capital market: \(w_K\) is given.
- Competitive cheesecake market: \(p\) is given.
Price-Taker Assumption

- Jack may influence the equilibrium price in reality.
  - Monopoly
  - Monopoly behavior
  - Oligopoly

Introduction

Short-Run Profit Maximization Problem
- Fixed Cost
- Short-Run Profit Maximization Problem
- Solution to Short-Run Profit Maximization Problem
- Isoprofit (What Jack Has to Select to Earn π)
- Feasibility (What Jack Can Select with His Tech)
- Tangency Condition
- Example

Comparative Statics

Long-Run Profit Maximization Problem

Returns to Scale & Profit Maximization Problem

Summary

Fixed Cost

- Long-run Jack maximizes his profit differently from short-run Jack.
  - Long-run Jack solves long-run profit maximization problem
  - Short-run Jack solves short-run profit maximization problem
Definition 2.1 (Short Run & Long Run)

1. A **short run** is a circumstance in which a firm is restricted in its choice of at least one input level.
2. A **long run** is the circumstance in which a firm is unrestricted in its choice of input levels.

**Fixed Cost**

Recall Definition 2.1 from Chapter 18:

- **Fixed cost** is a cost that Jack has to pay for the fixed input.
  - Jack has to pay the rent \( w_K X_K \) in the short run.
  - Suppose the size of kitchen if predetermined at \( X_K = 5 \).
  - \( FC = \)

- Fixed cost may or may not be a **sunk cost** (cost cannot be recouped, regardless of future actions) depending on the timing:
  - It is sunk after Jack signed up the lease.
  - Not if Jack hasn’t signed up the lease yet.
Short-Run Profit Maximization Problem

In the short run, Jack solves the short-run profit maximization problem (SPMP):

**Problem 2.3 (Short-Run Profit Maximization Problem (SPMP))**

Jack maximizes his short run profit given \( p, (\bar{w}_C, \bar{w}_K) \):

\[
\max_{x_C} \pi(x_C, \bar{x}_K) = \frac{pf(x_C, \bar{x}_K)}{w_C x_C - w_K \bar{x}_K} \quad \text{total revenue} - \text{total cost} = pf(x_C, \bar{x}_K) - FC.
\]

Solution to Short-Run Profit Maximization Problem

- **Key**: separate what Jack can earn from what Jack can produce for a while and patch them together later.
  - **Iso-profit line** tells what each production plan earns him.
  - **Feasible production plans** tell what he can actually produce.

- Short-run profit maximization problem is rather easy.
- Dumb or smart, Jack’s kitchen equipment is predetermined at \( \bar{x}_K \).
- All he has to do is to choose the right \( x_C \).
- i.e., he just needs to gage the relationship between \( y \) and \( x_C \).
- Everything else is not at his discretion \( (\bar{x}_K, p, \bar{w}_C, \bar{w}_K) \).
Isoprofit (What Jack Has to Select to Earn $\pi$)

- What is the relationship among cheesecake $y$, employment $x_C$ and profit $\pi$ then?
- Forget the production function for a while.
- As an accountant, tell Jack what combination of $x_C$ and $y$ (production plan) will generate $\pi$ be it feasible or not.

Definition 2.4 (Isoprofit Line)
An isoprofit line at $\pi$ contains all the production plans $(x_C, x_E, y)$ that.

- Once again, we are asking a hypothetical question here.
- How much profit can Jack earn if his production plan is $(x_C, x_E, y)$?
  - $(x_C, x_E, y)$ may not be technologically feasible.

Example 2.5 (Isoprofit Line)
Suppose $p = 4, w = (w_C, w_K) = (1, 1), \bar{x}_K = 10$. Find the isoprofit line at $\pi = 0$ and $\pi = 10$. 
Isoprofit (What Jack Has to Select to Earn $\pi$)

<table>
<thead>
<tr>
<th>$x_C$</th>
<th>$y$</th>
<th>$py$</th>
<th>$W_C x_C$</th>
<th>$W_C y$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.5</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>7.5</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>40</td>
<td>30</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>20</td>
<td>0</td>
<td>10</td>
<td>10</td>
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<tr>
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<tr>
<td>20</td>
<td>10</td>
<td>40</td>
<td>20</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>12.5</td>
<td>50</td>
<td>30</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

How do we read this graph?

1. Slice off along fixed $y$ or $x_C$.
   - Fix $y = 10$. The more $x_C$ is, the smaller the profit will be.
   - Fix $x_C = 10$. The more $y$ is, the larger the profit will be.
Proposition 2.6 (Slope of Isoprofit Line)
The slope of isoprofit line is given by \( \frac{w_C}{p} \).

Proof.
Rearrange \( \pi = py - w_Cx_C - w_Kx_K \).

Question:
If he hires one more chef, how many additional cheesecakes should he have to maintain his profitability?

Let’s double check:
\[
\pi(x_C = 20 + 1, \hat{x}_C = 10, y = 10 + \frac{1}{4}) = 4 \cdot \left(10 + \frac{1}{4}\right) - 1 \cdot (20 + 1) - 10 = \_._.
\]
Isoprofit (What Jack Has to Select to Earn $\pi$)

- Profit grows as he moves from southeast to northwest.
- As a profit maximizer, Jack wants to move to the northwesternmost point $(x_C, y) = (0, a \text{ lot})$
- ... but can he?

Feasibility (What Jack Can Select with His Tech)

- Recall p.16 from Chapter 18:

Example 2.7 (Feasible Production Plan)

Jack’s short-run production function is $f(x_C, x_K) = \sqrt{x_C x_K} = \sqrt{10 \sqrt{x_C}}$ when $x_K = 10$. His feasible production plan $(x_C, x_K, y)$ satisfies $y \leq f(x_C, x_K) = \sqrt{10} \sqrt{x_C}$.

Feasibility (What Jack Can Select with His Tech)

![Diagram of production function]

- Prod Fn

- Chef $x_C$

- Cheesecakes y (slices)

- 0 10 20 30 40 50 60 0.0 2.5 5.0 7.5 10.0 12.5 15.0 17.5 20.0 22.5 25.0
The slope of production function measures marginal product of a chef given $x_K = 10$.

Which production plan should Jack choose?
- On one hand, the northwestern, the better.
- On the other, any $y > f(x_C, x_K)$ is off limit.
Recall:
1. The slope of isoprofit line is $\frac{W_c}{P}$ (Proposition 2.6).
2. The slope of production function denotes the marginal product of $x_C$.  
3. Jack earns the maximum profit at $(x_C, \bar{x}_C, y)$ where the production function is just back to back with isoprofit.
**Tangency Condition**

**Condition 2.8 (Tangency Condition)**

At the optimal production plan \((x_C, \bar{x}_C, y)\),

\[
\frac{w_C}{p} = MP_C(x_C, \bar{x}_C).
\]
**Introduction**

**SPMP**

**Comp Stat**

**LPMP**

**Factor Dem’d**

**R2S**

**Tangency Condition**

1. **à la slices:**
   - LHS: Jack must have an increment of \( \frac{w_C}{p} \) slices to hire one more chef without hurting his profitability.
   - RHS: One more chef can produce \( MP_C(x_C, \bar{x}_C) \) slices with one more chef.
   - Market’s idea of chef’s worth (LHS) coincides with Jack’s idea of chef’s productivity (RHS) measured in terms of cheesecakes.

2. **à la dollars:**
   - LHS: Jack must pay \( w_C \) dollars to hire one more chef (marginal cost).
   - RHS: A new hire can produce \( MP_C(x_C, \bar{x}_C) \) slices, leading to \( pMP_C(x_C, \bar{x}_C) \) dollars of marginal revenue.
   - Market’s idea of chef’s worth (LHS) coincides with Jack’s idea of chef’s productivity (RHS) measured in terms of dollars.

3. **What if** \( w_C > pMP_C(x_C, \bar{x}_C) \)?
   - What if \( w_C < pMP_C(x_C, \bar{x}_C) \)?
Example 2.9 (Short-Run Profit Maximization Problem)

Suppose \( p = 4 \), \( w = (1, 1) \), \( x_C = 10 \) and \( y = f(x_C, x_K) = \sqrt{x_C \cdot x_K} \). Marginal product of chef is \( MP_C(x_C, x_K) = \frac{p}{\sqrt{x_C}} \). How many chefs should Jack hire?
Question 3.1 (Environmental Change & Hiring Decision)

How should Jack revise his hiring decision when

- $w_C \uparrow$
- $p \downarrow$?

Example 3.2 (Environmental Change & Hiring Decision)

Suppose $p = 4$, $w = (1, 1)$, $x_C = 10$ and $y = f(x_C, x_K) = \sqrt{x_C x_K}$. Marginal product of chef is $MP_C(x_C, x_K) = \frac{x_K}{2 \sqrt{x_C}}$. How many chefs should he hire or lay off when

- $w_C = 1/2$
- $p = 4 \times 2$?
Discussion 3.3 (Environmental Change & Hiring Decision)

Why does Jack have to downscale when \( w_C \uparrow \) or \( p \downarrow \)?

Example

Tangency condition (Condition 2.8):

\[
MP_C(x_C, \dot{x}_C) = \frac{w_C}{p}.
\]

- \( w_C \uparrow \) or \( p \downarrow \) increases the right-hand side.
- Marginal product is decreasing in \( x_C \) if Jack’s technology exhibits diminishing marginal product. 
- Jack has to stop hiring early on when each chef’s contribution is still high.

2 Cf. Definition 3.5 in Chapter 18.

Discussion 3.4 (Change in Fixed Cost)

- Does \( w_K \uparrow \) affect the optimal production plan \((x_C, \dot{x}_C, y)\)?
  - Does he hire more?
  - Does he produce more?
- Does \( \dot{x}_K \downarrow \) affect the optimal production plan \((x_C, \dot{x}_C, y)\)?
- Do they affect Jack’s decision making process similarly or differently?
Example 3.5 (Change in Fixed Cost)

Suppose \( p = 4, w = (1, 1), \bar{x}_F = 10 \) and 
\[ y = f(x_C, x_F) = \sqrt{x_C x_F}. \]

\[ MP_C(x_C, x_F) = \frac{\sqrt{x_C}}{2}. \]

How does the optimal production plan \((x^*_C, x^*_F, y^*)\) change when
- \( x_F = 10 \) and \( x_F \to 5 \).
- \( x_F = 10 \) and \( x_F \to 5 \).

**Graph:**
- Chef \( x_C \) vs. Cheesecakes \( y \) (slices)
Example

\[
\begin{align*}
\text{Chef}_C \times \text{Cheesecakes}_{y (slices)} &= \text{Isoprofit} \\
\text{Prod Fn} &= 0, 10, 20, 30, 40, 50, 60, 0.0, 2.5, 5.0, 7.5, 10.0, 12.5, 15.0, 17.5, 20.0, 22.5, 25.0, -55, -45, -35, -25, -15, -5, 5, 15, 25, 35, 45, 55, 65, 75, 85
\end{align*}
\]
Solution to Long-Run Profit Maximization Problem

Recall Definition 5.1 from Chapter 18:

**Definition 4.1 (Short Run & Long Run)**

1. A short run is a circumstance in which a firm is restricted in its choice of at least one input level.
2. A long run is a circumstance in which a firm is unrestricted in its choice of input levels.

Now Jack can change $x_K$ along with $x_C$.

---

Problem 4.2 (Long-Run Profit Maximization Problem (LPMP))

Given $w$ and $p$, long-run Jack solves

$$\max_{x_C, x_K} \pi(x_C, x_K) = \text{.}$$

The same condition Comment 2.8 applies to $x_K$:

$$\frac{w_K}{p} = MP_K(x_C^*, x_K).$$

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**Solution to Long-Run Profit Maximization Problem**

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Tangency Condition & MRTS

- Tangency conditions for long-run profit maximization problem:
  \[
  \frac{w_C}{p} = MP_C(x_C, x_k) \\
  \frac{w_K}{p} = MP_K(x_C, x_k).
  \]

- Then,
  \[
  \frac{w_C}{w_K} = \frac{MP_C(x_C, x_k)}{MP_K(x_C, x_k)} \\
  \Rightarrow \frac{w_C}{w_K} = MRTS(x_C, x_K). \quad (3)
  \]

3 Recall Definition 4.1 in Chapter 18.

- What does (3) mean?
- Wait for Chapter 20: Cost Minimization Problem.
Question 5.1 (Jack in the Labor Market)
Jack simultaneously appears in three markets:
- Cheesecake market (as a supplier)
- Labor market (as a buyer)
- Capital market (as a buyer)
How is Jack’s hiring decision reflected in labor demand?

Definition 5.2 (Factor Demand Function)
Factor demand function returns the optimal amount of input for each factor price given other parameters.

Factor demand is just another way to look at tangency condition (Condition 2.8):
\[
\frac{W_C}{p} = MP_C(x_C, \hat{x}_C).
\]
Tell Jack ongoing \(\frac{W_C}{p}\) and he will find \(x_C\) that satisfies Condition 2.8 above.
And that is factor demand.
We already know the answer (Question 3.1).
Example 5.3 (Factor Demand)

Suppose \( p = 4 \), \( w = (1, 1) \), \( x_K = 10 \) and
\[ y = f(x_C, x_K) = \sqrt{x_C x_K} \]. Marginal product of chef is
\[ MP_C(x_C, x_K) = \frac{1}{2} \sqrt{x_C} \]. How many chefs should he hire or lay off when
\[ w_C = 1/2? \]
Similarly, for other $w_C$’s, Jack determines his labor demand schedule according to Condition 2.8:

$$\frac{w_C}{p} = MP_x(x_C, x_C).$$

In Example 5.3, in particular, the factor demand is

$$\frac{\sqrt{x_C}}{2\sqrt{x_C}} = \frac{w_C}{p} \Rightarrow x_C = \frac{40}{w_C}.$$
Question 5.4 (Comparative Statics on Factor Demand)

What if $p$ plummeted in half?

Replace $p$ in Condition 2.8

$$\frac{w_C}{p} = MP_K(x_C, x_F).$$

with $p = 2$,

which leads to

$$\frac{\sqrt{x_C}}{2 \sqrt{x_C}} = \frac{w_C}{2} \Rightarrow x_C = \frac{10}{w_C}.$$
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Factor Demand

Returns to Scale & Profit Maximization Problem

IRS Will Explode

Summary

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How are returns to scale related to profit maximizing behavior?
Not all the technologies are compatible with profit maximization problem.

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Fact 6.1 (Returns to Scale & PMP)
If Jack is a price taker
- decreasing returns to scale: nice.
- increasing returns to scale: explosion if \( \pi > 0 \)
- constant returns to scale: nice if \( \pi = 0 \).
In what follows, consider a short-run profit maximization in which fixed cost is sunk.
IRS Will Explode

- CRS Jack is compatible with profit maximization problem only when
  \[ \frac{w_c}{\rho} = MP_C(x_c) \]
- and consequently raises zero profit.

IRS Will Explode

- Just because IRS Jack is not compatible with profit maximization problem does not mean he has to be a dumb Jack to find the optimal hiring and investment plan.
- Call for Chapter 20 Cost Minimization Problem.
Solving short-run and long-run profit maximization problem.

- Tangency condition:
  - Isoprofit vs feasible production plan.
- Comparative statics on PMP.
- Factor demand.
- Returns to scale compatible with profit maximization problem.

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