The Family of “Circle Limit III” Escher Patterns

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Abstract

The Dutch artist M.C. Escher created his four “Circle Limit” patterns which used the Poincaré model of hyperbolic geometry. Many people consider the third one of this sequence, *Circle Limit III* — a pattern of fish, to be the most beautiful. In this woodcut, four fish meet at right fin tips, three fish meet at left fin tips, and three fish meet at their noses. The backbones of the fish are aligned along white circular arcs. Fish on one arc are the same color, and all fish are colored according to the map-coloring principal: adjacent fish must have different colors.

We generalize Escher’s *Circle Limit III* pattern to an entire family of fish patterns. A pattern of fish in this family will have p fish meeting at right fins, q fish meeting at left fins, and r fish meeting at their noses meet. The number r must be odd so that the fish swim head to tail along backbone arcs. We denote such a pattern by the triple (p,q,r). *Circle Limit III* would be labeled (4,3,3) in this system. The pattern will be hyperbolic or Euclidean depending on whether $1/p + 1/q + 1/r$ is less than or equal to 1. Escher himself created a Euclidean pattern in this family — Notebook drawing 123, which we denote (3,3,3). Our Math Awareness Month 2003 poster pattern would be denoted (5,3,3). We will show more patterns of this family.

1. Introduction

Figures 1 and 2 below show computer renditions of M.C. Escher’s hyperbolic patterns *Circle Limit I* and *Circle Limit III* respectively. Escher was not satisfied with *Circle Limit I* but later rectified its failings in *Circle Limit III*. In a letter to the Canadian mathematician H.S.M. Coxeter, Escher wrote:

> *Circle Limit I*, being a first attempt, displays all sorts of shortcomings... There is no continuity, no “traffic flow,” nor unity of colour in each row... In the coloured woodcut *Circle Limit III*, the shortcomings of *Circle Limit I* are largely eliminated. We now have none but “through traffic” series, and all the fish belonging to one series have the same colour and swim after each other head to tail along a circular route from edge to edge... Four colours are needed so that each row can be in complete contrast to its surroundings. ([5], pp. 108-109, reprinted in [3])

Just as Escher was inspired to create his “Circle Limit” patterns by Coxeter, Coxeter, in turn was inspired to write two papers explaining the mathematics behind *Circle Limit III* [2, 3]. These papers also quote generously from the Escher-Coxeter correspondence. In the same issue of *The Mathematical Intelligencer* containing Coxeter’s second paper, an anonymous editor wrote the following caption for the cover of that issue, which showed Escher’s *Circle Limit III*:

> Coxeter’s enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet. ([1])

To my knowledge, Coxeter did not elucidate such a general theory. The goal of this paper is to provide such a theory. In order to develop this theory, we start with a short review of hyperbolic geometry and

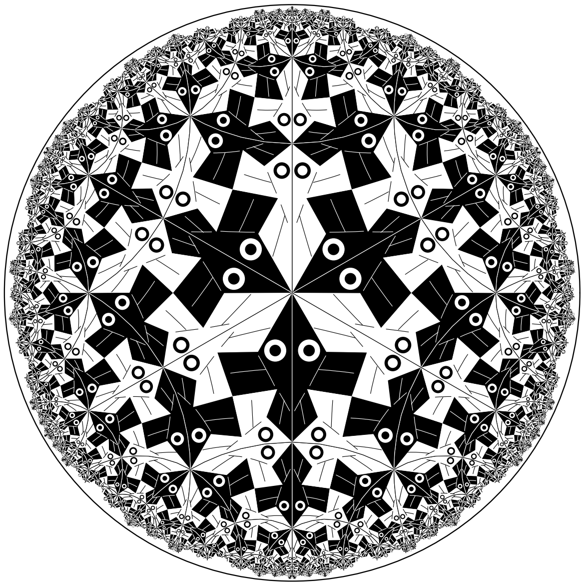

Figure 1: A rendition of Escher's *Circle Limit I*.

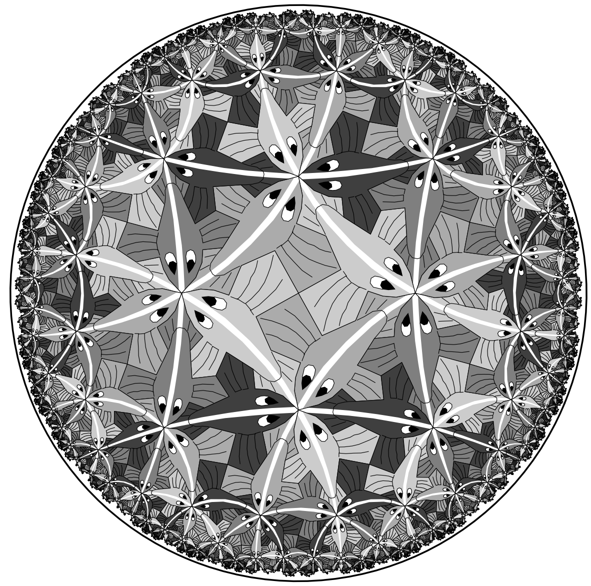

Figure 2: A rendition of Escher's *Circle Limit III*.

tessellations. With this background, we can present a general theory of "Circle Limit III" fish patterns. Finally we show several examples of these patterns, including an example of the special case in which the fish are symmetric.

2. Hyperbolic Geometry and Tessellations

Coxeter and Escher used the *Poincaré disk model* of hyperbolic geometry whose points are the interior points of a bounding circle, and whose (hyperbolic) lines are circular arcs orthogonal to the bounding circle (including diameters). The backbone lines and other features of the fish lie along hyperbolic lines in Figure 1. The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model (we say such a model is *conformal*), so all fish in a "Circle Limit III" pattern have roughly the same Euclidean shape. However, equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. For example, all the black fish in Figure 1 are hyperbolically the same size, as are the white fish; all the fish in Figure 2 are the same (hyperbolic) size.

It is tempting to guess that the backbone arcs of the fish in *Circle Limit III* are also hyperbolic lines, but this is not the case. These arcs are *equidistant curves*: they are an equal hyperbolic distance from the hyperbolic line with the same end points on the bounding circle; they are represented by circular arcs that intersect, but are not orthogonal to the bounding circle. For each hyperbolic line and a given distance, there are two equidistant curves, called *branches*, at that distance from the line, one one either side of the line. Escher only used one branch at at time in *Circle Limit III*. Equidistant curves are the hyperbolic analog of small circles in spherical geometry: a small circle of latitude in the northern hemisphere is equidistant from the equator (a great circle or "line" in spherical geometry), and has a corresponding small circle of latitude in the southern hemisphere the same distance from the equator.

There is a *regular tessellation*, $\{m, n\}$ of the hyperbolic plane by regular m-sided polygons meeting n at a vertex provided $(m-2)(n-2) > 4$. Figure 3 shows the regular tessellation $\{8, 3\}$ (heavy lines) superimposed on the *Circle Limit III* pattern. If one traverses edges of this tessellation, alternately going left, then right at each vertex, one obtains a zig-zagging path called a *Petrie polygon*. The midpoints of the edges of a Petrie polygon lie on a hyperbolic line by symmetry. The vertices of the Petrie polygon lie on one of two equidistant curve branches associated to that line — this is shown in Figure 4 with the Petrie polygon drawn with thick lines, the "midpoint" line and the equidistant curves drawn in a medium line, all superimposed on the $\{8, 3\}$ tessellation (lightest lines). These are the equidistant curves that Escher used.

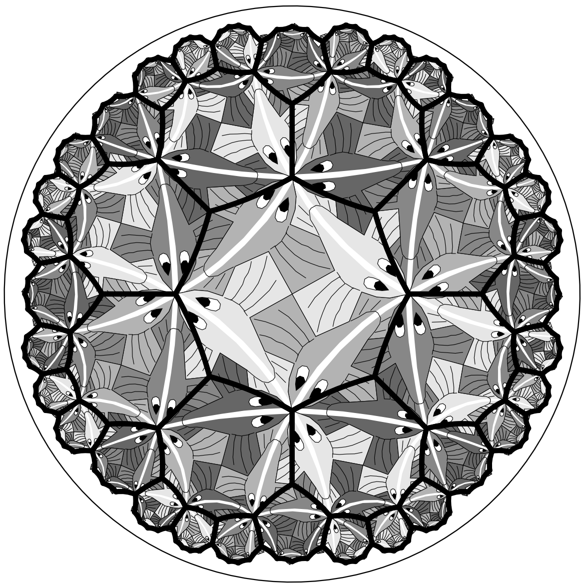

Figure 3: The tessellation $\{8, 3\}$ underlying the *Circle Limit III* pattern.

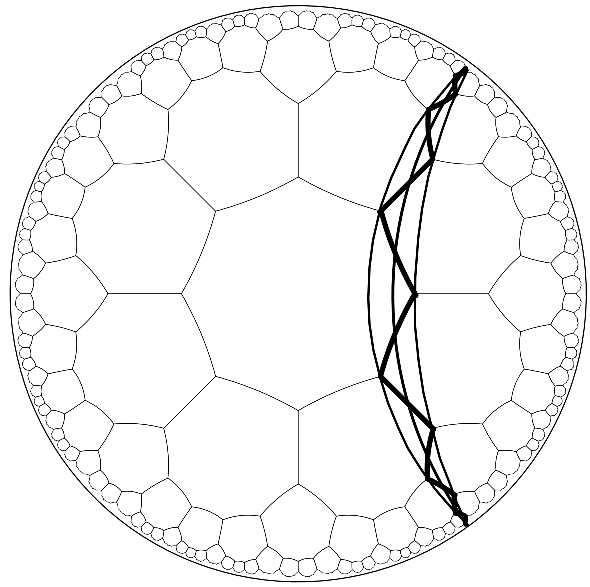

Figure 4: A Petrie polygon (heavy), a hyperbolic line and two equidistant curves (medium) associated to the $\{8, 3\}$ tessellation (lightest lines).

3. A General Theory of "Circle Limit III" Patterns

Looking at *Circle Limit III*, we see that 4 fish meet at right fin tips, 3 meet at left fin tips, and 3 meet at their noses. These are the numbers we want to generalize. We will show that there exist "Circle Limit III" patterns of fish in which p fish meet at right fins, q fish meet at left fins, and r fish meet at their noses. We denote such a pattern (p, q, r). *Circle Limit III* would be labeled $(4, 3, 3)$ in this notation.

We require r be odd so that the fish swim head-to-tail, in order to achieve "traffic flow." Also, by examining the left fins of *Circle Limit III*, we see that p and q should each be at least 3, since 2 fins could not have tips that meet. And of course r must be at least 3 too. Thus, the "smallest" example of such a pattern is $(3, 3, 3)$, which Escher realized in his *Notebook Drawing 123* page 216 of [6]. This pattern is based on the regular tessellation $\{3,3\}$ of the Euclidean plane by equilateral triangles, each triangle containing three half-fish. It is interesting that this simpler drawing is dated several years after the much more complex *Circle Limit III*. We also note that we do not consider *Notebook Drawing 122* to be a valid "Circle Limit III" pattern, since, as in *Circle Limit I*, fish meet "head-on", not head-to-tail. This pattern is based on the Euclidean tessellation of squares, with each square containing four half-fish. This pattern would be denoted $(4, 4, 2)$ if we allowed r to be even.

There is another tessellation that we can associate with *Circle Limit III*, obtained by dividing the octagons in Figure 3 into 4 "kites" — convex quadrilaterals with two pairs of adjacent equal sides. Each kite is a *fundamental region* for the pattern in that it contains exactly the right pieces of fish to assemble one complete fish. Figure 5 shows this kite tessellation superimposed on the *Circle Limit III* pattern. In general for the (p, q, r) pattern, one can use a kite-shaped fundamental region with vertex angles $\frac{2\pi}{p}$, $\frac{\pi}{r}$, $\frac{2\pi}{q}$, and $\frac{\pi}{r}$. A quadrilateral is hyperbolic precisely when its interior angle sum is less than 2π, which for our kites translates to the following inequality: $\frac{2\pi}{p} + \frac{\pi}{r} + \frac{2\pi}{q} + \frac{\pi}{r} < 2\pi$ or in other words: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Figure 6 shows the kite tessellation corresponding to the (3,3,5) pattern.

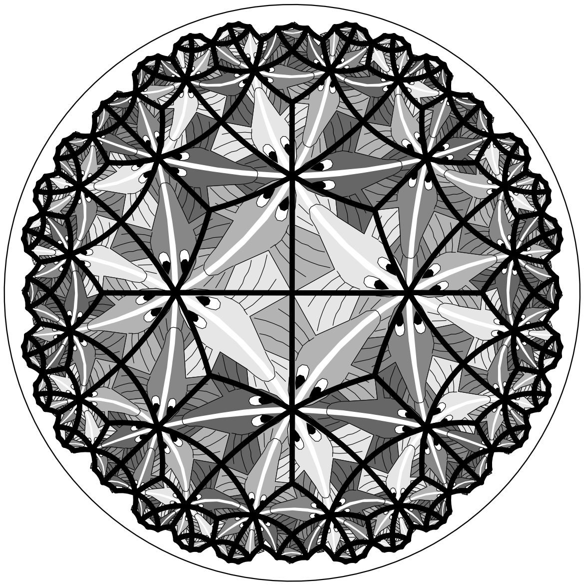

Figure 5: The kite tessellation superimposed on the *Circle Limit III* pattern.

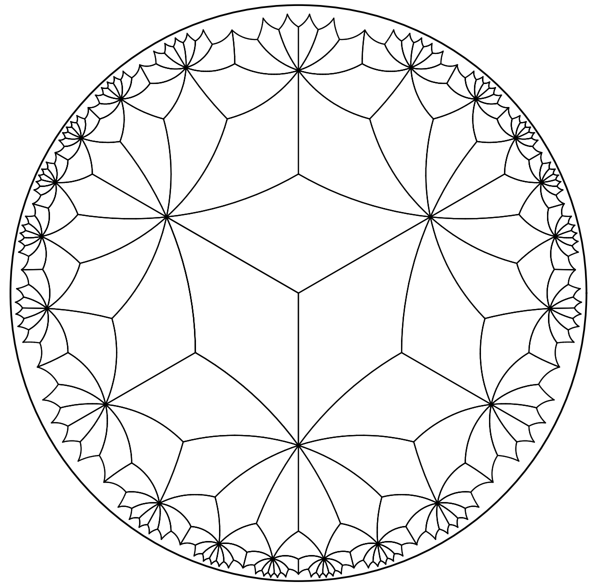

Figure 6: The kite tessellation corresponding to the (3,3,5) pattern.

4. Other "Circle Limit III" Patterns

The criticisms mentioned above that Escher had for his *Circle Limit I* pattern can largely be overcome by the following sequence of steps. First, two white fish meet at their noses, so it is impossible to have head-to-tail traffic flow. To fix this problem, we narrow the noses of the white fish so that a third one will fit in at each meeting point of noses. This solves the "traffic flow" problem, and we actually obtain a (4,4,3) "Circle Limit III" pattern of angular fish. To fix the "unity of colour" problem, it is easy to color any one line of fish the same color — the hard part comes in coloring all the lines of fish while not violating the map-coloring principle that adjacent fish should have different colors. In the spirit of Escher, the colors should be arranged symmetrically, using the smallest possible number of colors. This seems to be a very difficult problem in the general case. However, for a (4,4,3) "Circle Limit III" pattern, we only need to use three colors in order to do this. Figure 7 below shows how.

The patterns of Figures 6, 7, and Escher's Euclidean Notebook Drawing 123 are all examples of *symmetric* "Circle Limit III" patterns in which $p = q$. In this case the backbones fall on hyperbolic lines (or Euclidean lines in the case of Notebook Drawing 123). This is in contrast to the non-symmetric case in which $p \neq q$. In the non-symmetric case the fish must bend away slightly from the side with the most fish. Thus the fish in *Circle Limit III* bend to the left. In the non-symmetric case the backbones fall on equidistant curves to reflect this slight bend. As a final example, Figure 8 below shows the (5,3,3) pattern that was used for the 2003 Mathematics Awareness Month poster [4].

Professor Coxeter wrote two papers in which he calculated the intersection angle, ω, of the backbone lines of *Circle Limit III* with the bounding circle. In the first paper, he used hyperbolic trigonometry [2]. In the second paper, he used Euclidean techniques [3]. One of the calculations led to the equation $cos(\omega) = \sqrt{\frac{3\sqrt{2}-4}{8}}$. The value of ω is approximately $79.97°$. I carried out a similar calculation for the intersection

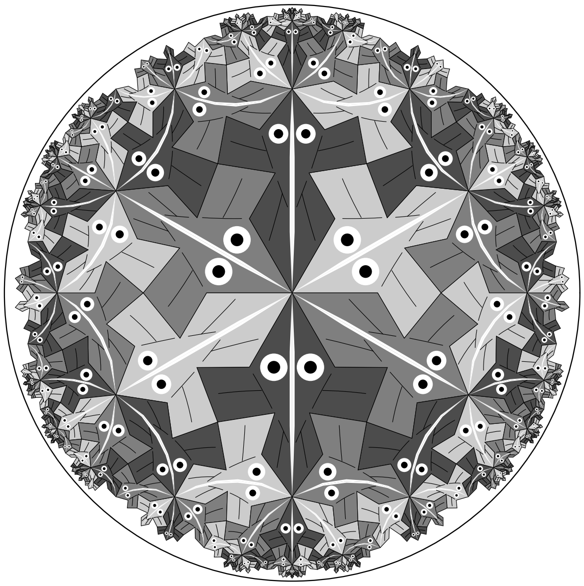

Figure 7: A (4,4,3) pattern in the style of *Circle Limit I*.

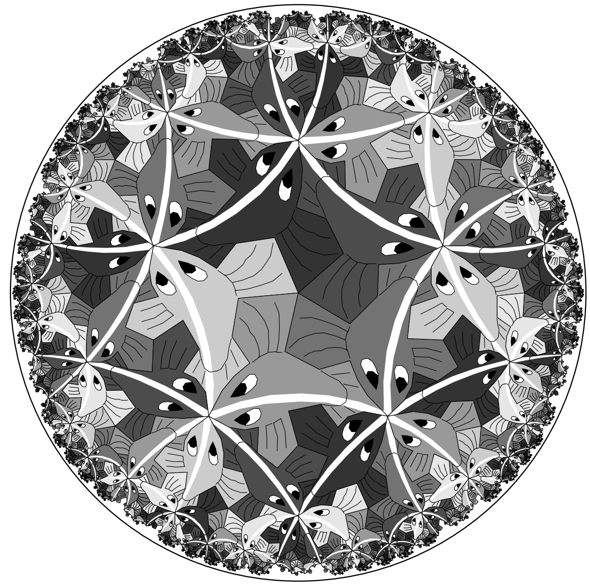

Figure 8: A (5,3,3) pattern.

angle of the pattern in Figure 8, the result of which was: $cos(\omega) = \sqrt{\frac{3\sqrt{5}-5}{40}}$. In this case, the value of ω is approximately $78.07°$.

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