Directions: Do all problems. Show all work. Make no mistakes.

1. Obtain the general solutions to the following differential equations. If initial conditions are given, **also** find the solution that satisfies the initial value problem. Answers need not be simplified, but integrals should be evaluated whenever you can. Implicit solutions will receive full credit only if labelled as such. Clearly indicate your final answer.

(a) (7 pts) $\frac{dP}{ds} = -5P, P(0) = 2.$

Inspection: $P(s) = P_0 e^{-5s}$

$P(0) = 2 \Rightarrow P_0 e^{-5\cdot 0} = 2$

i.e., $P_0 = 2 \Rightarrow \underline{P(s) = 2e^{-5s}}$

(3pts) Check your answer.

LHS $= P'(s) = 2(e^{-5s})\cdot(-5) = -10e^{-5s}$ $\qquad P(0) = 2e^{-5\cdot 0} = 2$ ✓

RHS $= -5P = -5\,2e^{-5s} = -10e^{-5s}$ (2) $\qquad$ (1)

RHS equals LHS ✓

(b) (10 pts) $\dot{y} - 3y = 2e^{3t}, y(0) = 1$

I.F. $\mu(t) = e^{\int -3dt} = e^{-3t}$

$\Rightarrow e^{-3t}\dot{y} - 3e^{-3t}y = 2e^{3t}\cdot e^{-3t} = 2$

i.e., $(e^{-3t}y)' = 2$

$\Rightarrow e^{-3t}y = 2t + C$

$y = 2te^{3t} + Ce^{3t}$

$y(0) = 0 + Ce^{3\cdot 0} = C = 1$

$\therefore \underline{y(t) = 2te^{3t} + e^{3t}}$

(5pts) Check your answer.

LHS $= \dot{y} - 3y = \frac{d}{dt}(2te^{3t} + 1\cdot e^{3t}) - 3(2te^{3t} + e^{3t})$

$= (2te^{3t}\cdot 3 + 2e^{3t} + 3e^{3t}) - 6te^{3t} - 3e^{3t}$

$= 6te^{3t} - 6te^{3t} + 5e^{3t} - 3e^{3t} = 2e^{3t}$ ✓

RHS $= 2e^{3t}$ ✓

$y(0) = 2\cdot 0\, e^{3\cdot 0} + e^{3\cdot 0} = 1$ ✓

(c) (10 pts) $y' = (3x^2 + 1)/y^3$, $y(0) = 2$.

$\Rightarrow y^3 \frac{dy}{dx} = 3x^2+1$

$\Rightarrow \int y^3 \frac{dy}{dx}\,dx = \int (3x^2+1)\,dx$

$\Rightarrow \int y^3\,dy = \int (3x^2+1)\,dx$

$\Rightarrow \underline{\frac{y^4}{4} = \frac{3x^3}{3} + x + C}$ (Implicit sln)

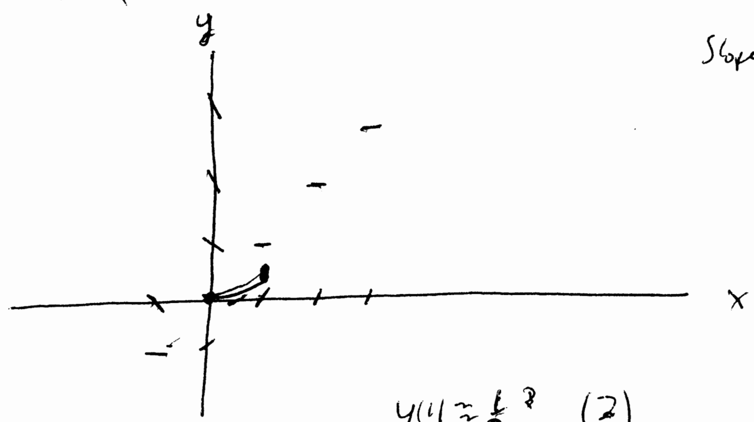
$y(0) = 2 \Rightarrow \frac{2^4}{4} = 0 + 0 + C \Rightarrow C = 4$

$\therefore \frac{y^4}{4} = x^3 + x + 4$ or $\underline{y^4 = 4x^3 + 4x + 16}$ (implicit sln)

or $\underline{y = (4x^3 + 4x + 16)^{\frac{1}{4}}}$

2. (10 pts) Sketch the portion of the slope field for $y'(x) = x - y$ in the first quadrant. Include slope marks on the (positive) axes. Include enough slope marks to allow you to sketch the solution corresponding to $y(0) = 0$. Use this sketch to estimate $y(1)$. (Do NOT solve analytically.) (2) (1) (2)

Slopes 5



$y(1) \approx \frac{1}{3}$? (2)

(or anything close to this)

3. (10 pts) Let $\phi(x)$ be the solution to the initial value problem $y' = x - y, y(0) = 0$. Approximate $\phi(1)$ using Euler's method with a step size of 0.5. You do not need to simplify your answers. Do not solve analytically.

$h = .25$

n	x_n	y_n	$f(x_n, y_n)$
0	0	0	$0 - 0 = 0$
1	.5	0	$.5 - 0 = .5$
2	1.0	.25	

$y_1 = y_0 + (.5) \cdot 0 = 0$

$y_2 = y_1 + (.5)(.5 - 0) = 0 + .25$

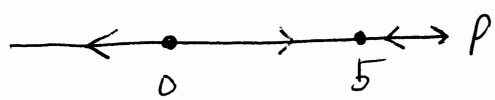
$= .25$

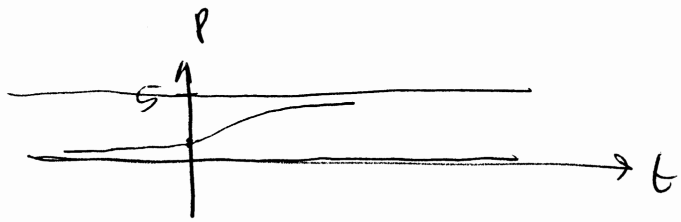
so, $\phi(1) \approx .25$

4. (10 pts) Sketch a phase line for the differential equation $\dot{P} = P(5 - P)$. Label your axis, indicate any equilibrium points with dots, and indicate the direction of growth/decline of the population (as time increases) with arrows. Using only the information from the phase line, sketch possible solutions corresponding to the initial conditions $P(0) = 1$ and $P(0) = 5$. Label your axes. Describe the long-term fate of the population $P(t)$ if $P(0) = 1$. (Do not solve the differential equation analytically.)

1/2 (5) (2) 1/2 (2)

$\dot{P} = P(5-P) = 0$ if $P = 0, 5$





$P(t) \to 5$

($P(t)$ increases, approaching 5.)

5. Consider the differential equation

$$y'(x) = 1 + \frac{y(x)}{x}.$$

(a) (10 pts) Make the substitution: $v(x) = \dfrac{y(x)}{x}$ to obtain a new differential equation in v and x (with y eliminated).

$\Rightarrow y = vx \Rightarrow y' = v + v'x$ (1)

$\therefore v + xv' = 1 + v$ (4)

$xv' = 1$

$\boxed{v' = \frac{1}{x}}$ (2)

$\Rightarrow v' = \dfrac{xy' - y}{x^2}$ (4)

$x^2 v' = xy' - y$

$xy' = x^2 v' + y = x + y$

$\Rightarrow x^2 v' = x$ (4)

$v' = \frac{1}{x}$ (2)

(b) (5 pts Extra Credit) Solve the new differential equation and use it to give a solution to the original differential equation.

$\Rightarrow v(x) = \ln|x| + C$ (3)

$\Rightarrow y(x) = x\ln|x| + Cx$ (2)