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Diff. Equations and Lin. Alg.
Math 3280 Quiz 7, Fall 2018
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1. Consider the following differential equation: $y'' - 6y' - 7y = 0$. Solve this differential equation by performing the following steps.

(a) (1 pt) Convert this second order differential equation into a system of first order equations.

Let $z_1 = y$, $z_2 = y'$

Then $y'' - 6y' - 7y$ becomes $z_2' - 6z_2 - 7z_1 = 0$

or, $\boxed{\begin{aligned} z_1' &= 0z_1 + 1z_2 \\ z_2' &= 7z_1 + 6z_2 \end{aligned}}$

(b) (1pt) Since the differential equation is linear, this system can be written in the form $\vec{x}' = A\vec{x}$, where A is a 2×2 matrix. Do this. Define A, and define $\vec{x}$ using notation you used in (a).

Let $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \vec{x}$; Let $A = \begin{bmatrix} 0 & 1 \\ 7 & 6 \end{bmatrix}$.

Then the system becomes $\vec{z}' = A\vec{z}$.

(c) (3pts) Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 0-\lambda & 1 \\ 7 & 6-\lambda \end{pmatrix} = (-\lambda)(6-\lambda) - (1 \cdot 7) = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$$
$$= 0 \text{ if } \lambda = 7 \text{ or } -1,$$

For $\lambda = 7$: $(A - 7I)\vec{v} = \vec{0}$ becomes

$$\left[\begin{array}{cc|c} -7 & 1 & 0 \\ 7 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -7 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $-7v_1 + v_2 = 0$ or $v_2 = 7v_1$. I choose $\vec{v} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

For $\lambda = -1$ $(A - \lambda I)\vec{v} = \vec{0}$ becomes

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 7 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $v_1 + v_2 = 0$. or $v_2 = -v_1$. I choose $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) (1pt) Use the eigenvalues and eigenvectors of A to write down the general solution to $\vec{x}' = A\vec{x}$ for your matrix A.

$$x(t) = c_1 e^{7t}\begin{bmatrix} 1 \\ 7 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. Assume the 2×2 matrix A has a complex eigenvalue $2 + 3i$ with a coresonding (complex) eigenvector $\begin{pmatrix} 5 - 2i \\ i \end{pmatrix}$. What is the general solution to the linear system of differential equations: $\vec{x}' = A\vec{x}$?

One complex solution is $e^{\lambda t}\vec{v} = e^{(2+3i)t}\begin{bmatrix} 5-2i \\ i \end{bmatrix} = (e^{2t}\cos 3t + ie^{2t}\sin 3t)\begin{bmatrix} 5-2i \\ i \end{bmatrix}$

$$= \begin{bmatrix} 5e^{2t}\cos 3t + 2e^{2t}\sin 3t \\ -e^{2t}\sin 3t \end{bmatrix} + i\begin{bmatrix} -2e^{2t}\cos 3t + 5e^{2t}\sin 3t \\ e^{2t}\cos 3t \end{bmatrix}$$

$\therefore$ gen'l sln is $c_1\begin{bmatrix} 5e^{2t}\cos 3t + 2e^{2t}\sin 3t \\ -e^{2t}\sin 3t \end{bmatrix} + c_2\begin{bmatrix} -2e^{2t}\cos 3t + 5e^{2t}\sin 3t \\ e^{2t}\cos 3t \end{bmatrix}$