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Diff. Equations and Lin. Alg.
Math 3280, B. Peckham
Quiz 9, Spring 2018

1. Condider the differential equation $y' - 4y = 5e^{4x}$.

   (a) (2 pts) Find the general solution using the first order linear technique by finding an integrating factor.

Use integrating factor $\rho(x) = e^{\int -4dx} = e^{-4x}$

$e^{-4x}y' - e^{-4x} \cdot 4y = 5e^{4x} \cdot e^{-4x} = 5$

ie, $(e^{-4x}y)' = 5$

Integrate: $e^{-4x}y = 5x + C$

$\Rightarrow y(x) = Ce^{4x} + 5xe^{4x}$

   (b) (4 pts) Solve using "guess" e^{rx} for y_c and method of annihilators to find a guess for y_p. Plug in the guess to determine y_p, and give the general solution.

For y_c: $y' - 4y = 0$. (1)

Try $y_c = e^{rx} \Rightarrow y_c' = re^{rx}$

$\Rightarrow (re^{rx}) - 4(e^{rx}) = 0$

ie, $e^{rx}(r-4) = 0$

$\Rightarrow r = 4$

$\therefore e^{4x}$ is one solution to (1)

$\Rightarrow$ all solutions are $y_c(x) = Ce^{4x}$

For y_p: Rewrite d.e. as $(D-4)[y] = 5e^{4x}$

Annihilate w/ $D-4$: $(D-4)(D-4)[y] = (D-4)[5e^{4x}] = 0$ (2)

$\Rightarrow r = 4$ is a dbl root for the characteristic polynomial for (2).

$\Rightarrow$ sln to (2) is $c_1e^{4x} + c_2xe^{4x}$

Throw out c_1e^{4x} since it is a solution to (1).

$\therefore$ Guess for y_p is $y_p = c_2xe^{4x}$

Plug in: $y_p' - 4y_p = 5e^{4x}$

ie, $c_2(4xe^{4x} + e^{4x}) - 4c_2xe^{4x} = 5e^{4x}$

ie, $c_2e^{4x} = 5e^{4x}$

$\Rightarrow c_2 = 5$

So $y_p(x) = 5xe^{4x}$

$\therefore$ Gen sln is $y(x) = y_c(x) + y_p(x) = Ce^{4x} + 5xe^{4x}$

2. (3 pts) Use USING GAUSSIAN ELIMINATION (row reduction operations) on the "augmented matrix" to convert it to row echelon form to find all solutions to the following system. Write your answer in vector form, and find a basis for the set of all solutions.

   $2x - 5y + z - w = 0$, $x + y - z + 5w = 0$.

$$\left[\begin{array}{cccc|c} 2 & -5 & 1 & -1 & 0 \\ 1 & 1 & -1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 5 & 0 \\ 2 & -5 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 5 & 0 \\ 0 & -7 & 3 & -11 & 0 \end{array}\right] \Rightarrow z, w \text{ free.}$$

Let $z = t$, $w = s$.

2nd eqn $\Rightarrow -7y + 3z - 11w = 0$

or $y = \frac{3z - 11w}{7} = \frac{3}{7}t - \frac{11}{7}w$

1st eqn $\Rightarrow x + y - z + 5w = 0$

$\Rightarrow x = -y + z - 5w = -\left(\frac{3}{7}t - \frac{11}{7}\right)w + t - 5s = \frac{4}{7}t - \frac{46}{7}s$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t\begin{bmatrix} 4/7 \\ 3/7 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{46}{7} \\ -\frac{11}{7} \\ 0 \\ 1 \end{bmatrix}$$

Basis is $\left\{ \begin{bmatrix} 4/7 \\ 3/7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -46/7 \\ -11/7 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. (3pts) Let $\mathcal{F}$ be the vector space of all functions from $\Re$ to $\Re$. Let $\mathcal{S}$ be the set of all solutions to the differential equation $y' - 4y = 0$. Show that $\mathcal{S}$ is a vector subspace or $\mathcal{F}$. Do this by showing the two "closure properties" of the solution set rather than explicitly finding solutions and showing that this set of solutions forms a vector subspace. (Give your answer on back.)

Method #1. Direct Pf w/o operator notation:

i. Let $f_1, f_2 \in S$. This means $f_1' - 4f_1 = 0$ and $f_2' - 4f_2 = 0$.

$\therefore (f_1+f_2)' - 4(f_1+f_2) = f_1' + f_2' - 4f_1 - 4f_2$

$= (f_1' - 4f_1) + (f_2' - 4f_2) = 0 + 0 = 0.$

$\therefore f_1 + f_2 \in S.$

ii Let $f_1 \in S$ and $c \in \mathbb{R}$. $f_1 \in S \Rightarrow f_1' - 4f_1 = 0$

$\therefore (cf_1)' - 4(cf_1) = cf_1' - c \cdot 4f_1$

$= c(f_1' - 4f_1) = c \cdot 0 = 0$

$\therefore cf_1 \in S.$

i and ii $\Rightarrow$ S is a subspace of $\mathcal{F}$. //

Method 2, using operator notation.

Let $L = D - 4$.

Then $f \in S \Rightarrow L[f] = 0$.

i. Let $f_1, f_2 \in S$. this means $L[f_1] = 0$ and $L[f_2] = 0$

$\therefore L[f_1 + f_2] = L[f_1] + L[f_2] = 0 + 0 = 0$

$\therefore f_1 + f_2 \in S$

ii Let $f_1 \in S$ and $c \in \mathbb{R}$. $f_1 \in S \rightarrow L[f_1] = 0$

$\therefore L[cf_1] = cL[f_1] = c \cdot 0 = 0$

$\therefore cf_1 \in S.$

i and ii $\Rightarrow$ S is a subspace of $\mathcal{F}$. //