Math 3280
Differential Equations with Linear Algebra

Test 1

B. Peckham

Sept. 21, 2018

Name A.K.

SHOW ALL WORK.

Calculators may be used for algebra and graphing only. You may NOT use calculators to solve differential equations (using commands like DSolve) or to take integrals or derivatives (using commands like D or Integrate).

p2.________/32+4

p3.________/20

p4.________/34

p5.________/14+6

Total________/100+10

Directions: Do all problems. Show all work. Make no mistakes. Calculators may be used only for algebra and graphing, but not for symbolic tasks like integrating, differentiating, or solving differential equations. Implicit solutions will receive full credit only if labelled as **implicit**.

1. Find both the general solution to the given differential equations, and the specific solution corresponding to the given initial conditions. Show your work.

(a) (6 pts) $\frac{dy}{ds} = 2s$, $y(0) = 3$

$y = \int 2s\, ds + C = s^2 + C$

$y(0) = 2 \Rightarrow 0^2 + C = 3$

$\Rightarrow C = 3$

$\therefore y(s) = s^2 + 3$

(b) (6 pts) $\frac{dy}{ds} = 2y$, $y(0) = 3$

"$y' = ay$" $\Rightarrow$ by inspection

$y(s) = Ce^{2s}$

$y(0) = 2 \Rightarrow Ce^{2\cdot 0} = C = 2$

$\therefore y(s) = 3e^{2s}$

(c) (10 pts) $\dot{y} - 2ty = 3e^{t^2}$, $y(0) = 2$. Show all steps; don't just "plug into" a formula.

1st order linear: $p(t) = -2t \Rightarrow \rho(t) = e^{\int -2t\, dt} = e^{-t^2}$

Multiply by e^{-t^2}: $\dot{y}e^{-t^2} - 2tye^{-t^2} = 3e^{t^2}e^{-t^2}$

i.e., $\left(ye^{-t^2}\right)' = 3 \cdot 1$

$\Rightarrow ye^{-t^2} = \int 3\, dt + C = 3t + C$

$\therefore y(t) = 3te^{t^2} + Ce^{t^2}$

$y(0) = 1 \Rightarrow$

$3 \cdot 0 \cdot e^{0^2} + Ce^{0^2} = 2$

$\Rightarrow C = 2$

$\therefore y(t) = 3te^{t^2} + 2e^{t^2}$

(d) (10 pts) $y' = (2x+1)/y^2$, $y(1) = 2$.

Separable: $y^2 y' = (2x+1)$

$\Rightarrow \int y^2 \frac{dy}{dx}\, dx = \int 2x+1\, dx + C$

i.e., $\int y^2\, dy = \frac{2x^2}{2} + x + C$

$\frac{y^3}{3} = \frac{\cancel{2}x^2}{\cancel{2}} + x + C$ (implicit)

or $y = (3x^2 + 3x + \tilde{C})^{\frac{1}{3}}$ (explicit)

$y(1) = 3 \Rightarrow$

$\frac{3^3}{3} = \frac{\cancel{2}1^2}{\cancel{2}} + 1 + C$

$\Rightarrow 9 = 2 + C$

or $C = 7$

$\therefore \frac{y^3}{3} = x^2 + x + 7$ (implicit)

or $y = (3x^2 + 3x + 21)^{\frac{1}{3}}$ (explicit)

(e) (4 pts Extra Credit) What is the interval of existence for the solution to the initial value problem in (d)?

$x \in (-\infty, \infty)$

2. (8 pts) For what values of r is e^{rx} a solution to $y''(x) - 3y'(x) + 2y(x) = 0$?

Try $y(x) = e^{rx} \Rightarrow y' = re^{rx}$, $y'' = r^2 e^{rx}$

Plug in: $r^2 e^{rx} - 3re^{rx} + 2e^{rx} = 0$

$\Rightarrow e^{rx}(r^2 - 3r + 2) = 0 \Rightarrow e^{rx}(r-1)(r-2) = 0$

$\Rightarrow \underline{r = 1, 2}$

(So e^x, e^{2x} are solutions)

(We will see later that any function of the form $C_1 e^x + C_2 e^{2x}$ will be a solution)

3. Consider the initial value problem (differential equation and initial condition):

$$y' = (\frac{y}{x^2})^2 + 3\frac{x^2}{y} + 2\frac{y}{x} \quad ; \quad y(2) = 1$$

(a) (8 pts) Obtain a differential equation and initial condition for v by using the substitution $y = x^2 v$ to eliminate y and obtain a new differential equation. Do not solve!

$y = x^2 v \Rightarrow y' = x^2 v' + 2xv$

Substitute: $x^2 v' + 2x\cancel{y} = v^2 + \frac{3}{v} + 2\frac{x^2 v}{x}$

$\Rightarrow x^2 v' + 2xv = v^2 + \frac{3}{v} + 2xv$

$\therefore x^2 v' = v^2 + \frac{3}{v}$

$\Rightarrow \underline{v' = \left(v^2 + \frac{3}{v}\right)\frac{1}{x^2}}$

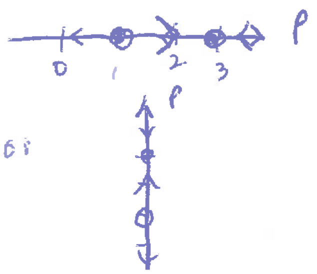
$y(2) = 1 \Rightarrow 1 = 2^2 \cdot v(2) \Rightarrow \underline{v(2) = \frac{1}{4}}$

(b) (2 pts) Is the original differential equation separable, linear, neither or both?

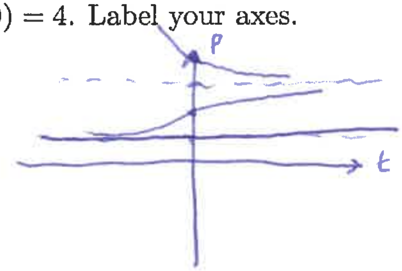
(c) (2 pts) Is the new differential equation separable, linear, neither or both?

4. Consider the differential equation $\dot{P} = (1-P)(P-3)$.

   (a) (6 pts) Sketch a phase line for the differential equation. Label your axis, indicate any equilibrium points with dots, and indicate the direction of growth/decline of the population (as time t increases) with arrows.



   (b) (6 pts) Using only the information from the phase line, sketch possible solutions (both forward and backward in time) corresponding to the initial conditions $P(0)=1$, $P(0) = 2$ and $P(0) = 4$. Label your axes.

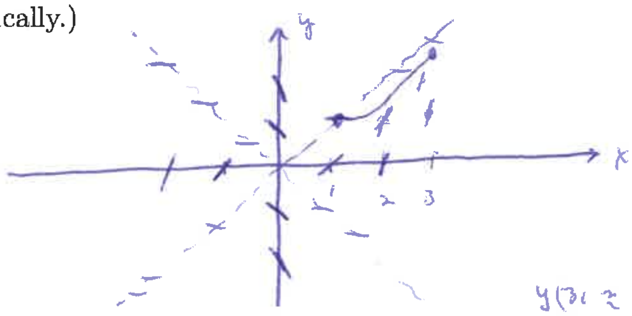


   (c) (2 pts) Describe the long-term (forward) fate of the population P for ~~both~~ all these initial conditions. (Do not solve the differential equation analytically.)

   $P(t) \to 3$ for $P(0) = 2$ or 4

   $P(t) = 1$ $\forall t$ for $P(0) = 1$.

5. (10 pts) Sketch the portion of the slope field for $y'(x) = x^2 - (y(x))^2$ along both axes and along the lines $y = x$ and $y = -x$. Include enough additional slope marks to allow you to sketch the solution corresponding to $y(1) = 1$. Use this sketch to estimate $y(3)$. (Do NOT solve analytically.)



On x axis

x	y	y'
-2	0	4
-1	0	1
0	0	0
1	0	1
2	0	4

x	y	y'
2	1	3
3	1	8
3	2	5

$y(3) \approx 2.5$?

6. (10 pts) Let $\phi(x)$ be the solution to the initial value problem $y'(x) = x^2 - (y(x))^2, y(1) = 1$. Numerically approximate $\phi(3.0)$ using Euler's method with a step size of 1.0. You do not need to simplify your answers. Do not solve analytically.

n	x_n	y_n	y_n'
0	1	1	0
1	2	1	3
2	3	4	

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$

$y_1 = 1 + 1 \cdot 0 = 1$

$y_2 = 1 + 1 \cdot 3 = 4$

$y(3) \approx 4$.

7. (8 pts) Assume that an island population has 10,000 inhabitants, 50 of whom have just contracted the dreaded mahtimahtica disease. Assume the rate at which inhabitants contract this disease is proportional to the product of the number of inhabitants with the disease and the number without. Assume those with the disease are cured at a rate proportional to the number who are diseased. No inhabitants come to or from the island. No inhabitants die. ~~Write a differential equation and corresponding initial conditions which would serve as a model for the diseased population.~~ Call $D(t)$ the diseased population at any time t. Write an initial value problem (differential equation and initial conditions) to model this population so that any parameters in the differential equation are positive. (Do **not** solve.)

$$\dot{D}(t) = k_1 D(10\,000 - D) - k_2 D, \qquad D(0) = 50$$

8. The textbook derives a model for the velocity of parachutist falling as:

$$\frac{dv}{dt} = -g - \rho v.$$

The book then states that the solution to this differential equation is $v(t) = (v_0 + \frac{g}{\rho})e^{-\rho t} - \frac{g}{\rho}$. where $v_0 = v(0)$

(a) (6 pts) Verify "by plugging in" that the given function for $v(t)$ is a solution to the differential equation.

$$LHS = v'(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t}\cdot(-\rho) - 0 = -v_0\rho e^{-\rho t} - g e^{-\rho t}$$

$$RHS = -g - \rho v = -g - \rho\left(\left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}\right)$$

$$= -g - v_0\rho e^{-\rho t} - g e^{-\rho t} + g$$

$$= -v_0\rho e^{-\rho t} - g e^{-\rho t}$$

LHS = RHS, ∴ $v(t)$ is a sln.

(b) (6pts EC) Use the first order linear solution technique to show how you would arrive at the solution. Show all steps.

$v' = -g - \rho v \iff v' + \rho v = -g$

1st order lin. ⇒ Int factor is $e^{\int \rho\, dt} = e^{\rho t}$

Multiply: $e^{\rho t} v' + e^{\rho t}\rho v = -g e^{\rho t}$

Rewrite $(e^{\rho t} v)' = -g e^{\rho t}$

Integrate: $e^{\rho t} v = \int -g e^{\rho t}\, dt + C$

$= -g\frac{e^{\rho t}}{\rho} + C$

Mult. by $e^{-\rho t}$:

$v(t) = -\frac{g}{\rho} e^{\rho t} e^{-\rho t} + C e^{-\rho t}$

$= -\frac{g}{\rho} + C e^{-\rho t}$

$v(0) = v_0 \Rightarrow -\frac{g}{\rho} + C e^{0} = v_0$

$\Rightarrow -\frac{g}{\rho} + C = v_0$

or $C = \left(v_0 + \frac{g}{\rho}\right)$

∴ $v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}$