Math 3280

Differential Equations with Linear Algebra

Test 1

B. Peckham

~~Oct.~~ Feb. ~~3~~ 13, ~~2014~~ 2015

Name A.K.____

SHOW ALL WORK.

Calculators may be used for algebra and graphing only. You may NOT use calculators to solve differential equations (using commands like DSolve) or to take integrals or derivatives (using commands like D or Integrate).

p2.________/26

p3.________/30

p4.________/20

p5.________/24

EC.________/5

Total________/75

Directions: Do all problems. Show all work. Make no mistakes. Calculators may be used only for algebra and graphing, but not for symbolic tasks like integrating, differentiating, or solving differential equations. Implicit solutions will receive full credit only if labelled as **implicit.**

1. (4 pts) Consider the following differential equations. Are they separable, linear, both or neither? (Do not solve.)

   (a) $\dot{P} = 2P$ $\quad \dot{P} - 2P = 0$ lin. $\quad \frac{dP}{P} = 2\,dt$ sep

   (b) $\dot{P} = 2t$ $\quad \dot{P} - 0P = 2t$ lin $\quad dP = 2t\,dt$ sep

2. ~~(8 pts)~~ Solve the following two initial value problems. Explain your work.

   (a) (6 pts) $\dot{P} = 2P$, $P(0) = 2$ $\quad$ Inspection: $P(t) = P_0 e^{2t}$ $\quad P(0) = 2 \Rightarrow P(t) = 2e^{2t}$

   5 $\quad$ 1

   (b) (6 pts) $\dot{P} = 2t$, $P(0) = 2$ $\quad$ Integrate wrt t $\quad P(t) = \frac{2t^2}{2} + C = t^2 + C$ 5

   $P(0) = 3 \Rightarrow C = 2 \Rightarrow P(t) = t^2 + 2$

   1

3. (10 pts) Find the general solution to $\dot{y} + 3y = 2e^{3t}$ and the specific solution when you add the initial condition $y(0) = 1$.

Linear (1st order) Let $\rho(t) = e^{\int 3dt} = e^{3t}$

$\dot{y}e^{3t} + 3e^{3t}y = 2e^{3t}e^{3t} = 2e^{6t}$

$(ye^{3t})' = 2e^{6t}$

Int w.r.t. t: $ye^{3t} = \frac{2e^{6t}}{6} + C$

Explicit: $y(t) = \frac{e^{6t}}{3} \cdot e^{-3t} + Ce^{-3t}$

i.e., $y(t) = \frac{1}{3}e^{3t} + Ce^{-3t}$ (Gen Sln)

$y(0) = 1 \Rightarrow$

$\frac{1}{3} + C = 1 \Rightarrow C = \frac{2}{3}$

2 $\quad \therefore y(t) = \frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t}$

(4 pts) Verify that your answer satisfies both the differential equation and the initial condition.

$y(t) = \frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t} \Rightarrow \dot{y}(t) = \frac{1}{3}3e^{3t} + \frac{2}{3}(-3)e^{-3t}$

$= e^{3t} - 2e^{-3t}$

3

So $\dot{y} + 3y = (e^{3t} - 2e^{-3t}) + 3\left(\frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t}\right)$

$= 2e^{3t} + \left(-2 + \frac{2}{3} \cdot 3\right)e^{-3t} = 2e^{3t}$ ✓

1 $\quad$ IC: $y(0) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 1 = 1$ ✓

4. (10 pts) Find the general solution to $y' = (2x+1)/y$ and the specific solution when you add the initial condition $y(1) = 0$. $\boxed{y(2) = \sqrt{8}}$

$y \frac{dy}{dx} = (2x+1)\, dx$

$y(1) = 0 \Rightarrow \frac{0^2}{2} = 1^2 + 1 + C \Rightarrow C = -2$

$\therefore\ y(x) = \sqrt{2}\left(\pm\sqrt{x^2+x+C}\right)$ explicit general sln

$\Rightarrow \int y \frac{dy}{dx}\, dx = \int (2x+1)\, dx$

$\boxed{\frac{y^2}{2} = x^2 + x + C}$ General Implicit

$y(1)=0 \Rightarrow \frac{y^2}{2} = x^2 + x - 2$ Specific Implicit

$\Rightarrow y = \pm\sqrt{2x^2+2x-4} = \sqrt{2}\left(\pm\sqrt{x^2+x-2}\right)$

(5 pts Extra Credit) What is the interval of existence for this initial value problem above?

Poorly Worded Problem. Ignore.

Need $x^2+x-2 \geq 0$, so

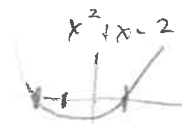
$= (x+2)(x-1)$

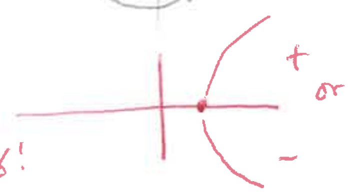
$\Rightarrow$ problems if $-2 < x < 1$

$\therefore\ x > 1$

But $y'(1)$ dne $\therefore$ int $= \emptyset$!

x^2+x-2





5. (10 pts) For what values of A and r is Ae^{rx} a solution to $y''(x) - 3y'(x) + 2y(x) = 0$?

$y = Ae^{rx} \Rightarrow y' = rAe^{rx},\ y'' = r^2Ae^{rx}$ 2

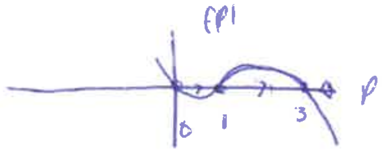
LHS: $Ae^{rx}(r^2 - 3r + 2) = 0 \Leftrightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$

2 2

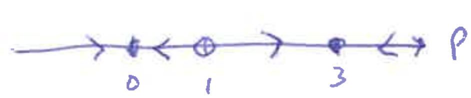
$\Rightarrow r = 1, r = 2$ 2

A anything 2

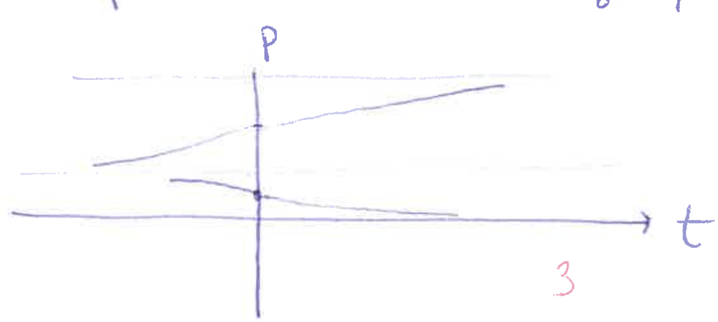
6. (10 pts) Sketch a phase line for the differential equation $\dot{P} = P(1-P)(P-3)$. Label your axis, indicate any equilibrium points with dots, and indicate the direction of growth/decline of the population (as time t increases) with arrows. Using only the information from the phase line, sketch possible solutions (both forward and backward in time) corresponding to the initial conditions $P(0) = 0.5$ and $P(0) = 2$. Label your axes. Describe the long-term (forward) fate of the population P for both initial conditions. (Do not solve the differential equation analytically.)



Phase line



5

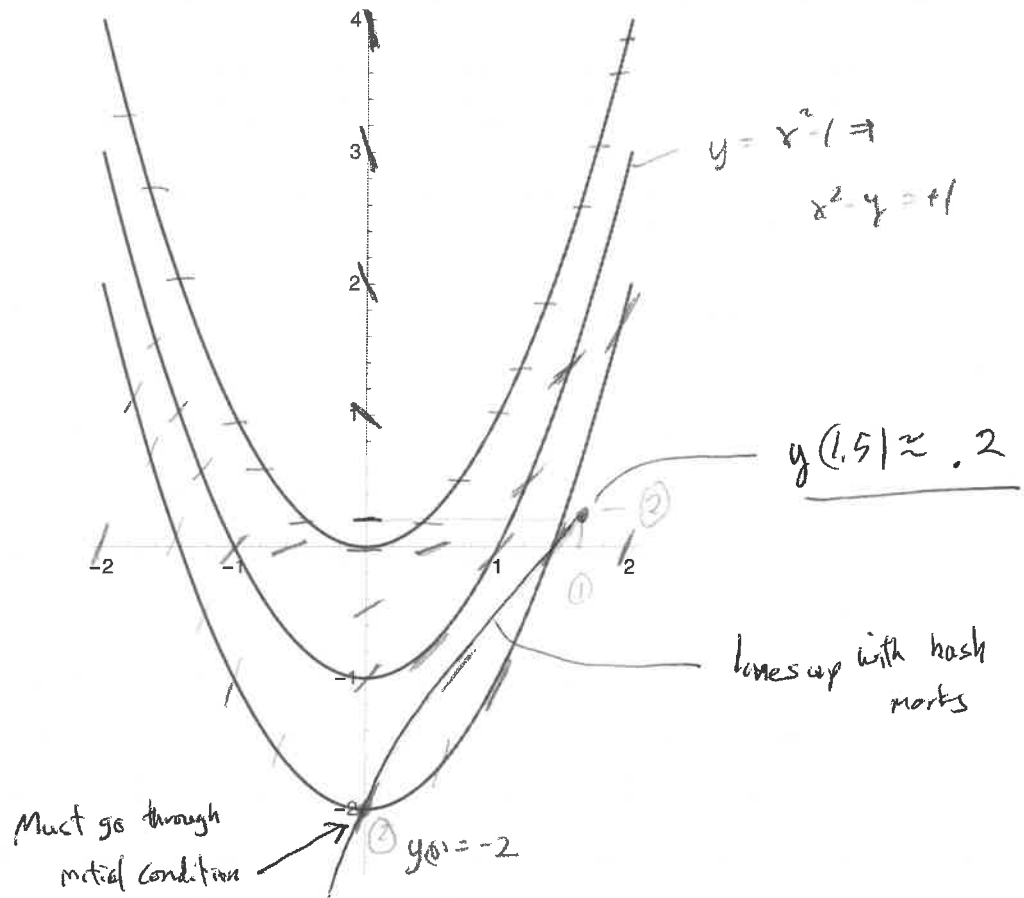


$P(0) = 2 \Rightarrow P(t) \to 3$ 1

$P(0) = .5 \Rightarrow P(t) \to 0$ 1

3

7. (10 pts) Sketch the portion of the slope field for $y'(x) = x^2 - y(x)$ along both axes and along the parabolas $y = x^2$, $y = x^2 - 1$ and $y = x^2 - 2$. (The three parabolas are shown for reference.) Include enough additional slope marks to allow you to sketch the solution corresponding to $y(0) = -2$. Use this sketch to estimate $y(1.5)$. (Do NOT solve analytically.)



8. (10 pts) Let $\phi(x)$ be the solution to the initial value problem $y'(x) = x^2 - y(x), y(0) = -2$. Numerically approximate $\phi(1.5)$ using Euler's method with a step size of 0.5. You do not need to simplify your answers. Do not solve analytically.

n	x_n	y_n
0	0	-2
1	.5	-1
2	1.0	$-.375$
3	1.5	$\boxed{.3125}$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$= -2 + (.5) \cdot (0 - (-2))$

$= -2 + 1 = -1$

$y_2 = -1 + (.5)((.5^2) - (-1))$

$= -1 + \frac{1.25}{2} = -1 + .625$

$= -.375$

$y_3 = -.375 + (.5)(1^2 + .375)$

$= -.375 + \frac{1.375}{2} = .3125$

9. Consider the differential equation $y' + y = \frac{1}{y^2}$. Make the substitution $v = y^3$ to replace y with v. Keep the independent variable as x. (8 pts) What is the differential equation in the new variables? (2 pts) Is this substitution useful in solving the original differential equation? Justify.

$v = y^3 \Rightarrow v' = 3y^2y'$ 3

$3y^2 \cdot y' + y \cdot 3y^2 = 3y^2 \frac{1}{y^2}$ 3

8

Sub: $v' + 3v = 3$ 2

2 Yes: New eqn is linear, Orig. was non lin (But Sep !:)

10. Knowing that the rate of heating or cooling of hot coffee or cold icewater is proportional to the temperature difference between the drink and the (constant) room temperature T_0 leads to the following differential equation: $\dot{T} = k(T - T_0)$.

(a) (3pts) If the drink is coffee (hot), is $k > 0$ or $k < 0$? Explain briefly.

$\dot{T} < 0$, $T - T_0 > 0$ so $k < 0$

(b) (3pts) If the drink is icewater (cold), is $k > 0$ or $k < 0$? Explain briefly.

$\dot{T} > 0$, $T - T_0 < 0$, so $k < 0$

11. (10 pts) Assume that if unchecked, the population of mathematicians would grow at a rate proportional to the number of mathematicians alive at any given time. Assume, however, that there are two checks are operating. First, because of the interfering brain waves when too many mathematicians are too close together, mathematicians are eliminated at rate proportional to their population squared. (This is independent of the effect too many mathematicians have on the rest of humanity.) Secondly, in an effort to control the population of mathematicians the government has decided to eliminate them at a rate of 100 per year. If there are currently 10,000 living mathematicians, write a differential equation and initial condition, that describe the change in population as a function of time. Label any variables and constants you use; distinguish between them; indicate whether the constants should be positive or negative. (Do not solve.)

1 Let ~~$M(t)$~~ = # mathematicians alive @ time t.

$\dot{M} = k_1 M - k_2 M^2 - 100$, $M(0) = 10000$

1 $k_1 > 0$ $k_2 > 0$ 2

2 2 2

Math 3280
Differential Equations with Linear Algebra

Test 2
B. Peckham
October 23, 2018

Name A. K.

Directions: Do all problems. Make no mistakes. SHOW ALL WORK. Closed book. Calculators may be used for algebraic computations, but not for solving differential equations or doing row reduction.

2.________/24

3.________/34 25

4.________/26+1 25

5.________/16

EC________/+6

Total________/100+6

1. Consider the differential equation $y'' + y' - 6y = 0$.

(a) (8 pts) Find the general solution by guessing solutions of the form $y = e^{rx}$. Show your work from this guess.

$y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$

Plug in: $r^2 e^{rx} + re^{rx} - 6e^{rx} = 0$

factor $e^{rx}(r^2 + r - 6) = 0$

$\Rightarrow r^2 + r - 6 = 0 \quad (e^{rx} \text{ never } = 0)$

$\Rightarrow (r+3)(r-2) = 0 \Rightarrow r = -3, 2$

$\therefore e^{-3x}, e^{2x}$ are 2 solutions

$\Rightarrow c_1 e^{-3x} + c_2 e^{2x}$ is the general solution

(b) (6 pts) Find one solution to the related nonhomogeneous differential equation: $y'' + y' - 6y = 3x$ by guessing a soution of the form $y = Ax + B$.

$y = Ax + B \Rightarrow y' = A \Rightarrow y'' = 0$

Plug in: $0 + A - 6(Ax + B) = 3x$

ie, $A - 6B - 6Ax = 0 + 3x$

$\therefore -6A = 3,\ A - 6B = 0 \Rightarrow A = -\frac{3}{6} = -\frac{1}{2}$

$\Rightarrow -\frac{1}{2} - 6B = 0$

$\Rightarrow 6B = -\frac{1}{2}$

$\Rightarrow B = -\frac{1}{12}$

$\therefore y = -\frac{1}{2}x - \frac{1}{12}$ is one solution

(c) (2pts) Use (a) and (b) to determine the general solution to $y'' + y' - 6y = 3x$? If you did not answer (a) or (b), indicate how you would use those answers to determine the answer to this problem.

$y(x) = c_1 e^{-3x} + c_2 e^{2x} + \left(-\frac{1}{2}x - \frac{1}{12}\right)$

2. Consider the differential equation $y'' + 4y = \cancel{32}\,24e^{2x}$. One solution to this differential equation is $y_p(x) = 3e^{2x}$. The complementary solution, to $y'' + 4y = 0$, is $y_c(x) = c_1\cos(2x) + c_2\sin(2x)$.

(a) (2 pts) What is the general solution to $y'' + 4y = \cancel{32}\,24e^{2x}$?

$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + 3e^{2x}$

(b) (6 pts) What is the solution to $y'' + 4y = \cancel{32}\,24e^{2x}$ that also satisfies the initial conditions $y(0) = 2, y'(0) = 0$?

(a) $\Rightarrow y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x) + 6e^{2x}$

So $y(0) = c_1 + c_2 \cdot 0 + 3 \cdot 1 = 2$

$y'(0) = 0c_1 + 2c_2 + 6 \cdot 1 = 0$

ie,

$1c_1 + 0c_2 = -1$

$0c_1 + 2c_2 = -6$

$\Rightarrow c_2 = -3$

$c_1 = -1$

$\therefore y(x) = -\cos(2x) - 3\sin 2x + 3e^{2x}$

3. (6 pts) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Define a matrix B so that $BA = \begin{bmatrix} a_{11} - 2a_{21} & a_{12} - 2a_{22} & a_{13} - 2a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$BA = \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A}$$

4. (3 pts) If A is a 3×3 matrix, and $\det(A) = 5$, what is $\det(2A)$? Explain briefly.

$$\det(2A) = 2^3 \cdot \det(A)$$

5. (8 pts) Solve the following linear system USING GAUSSIAN ELIMINATION (row reduction to echelon or reduced echelon form). Leave your answers as exact fractions - not calculator approximations.

$$\begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 2 & -1 & 1 \\ 4 & 5 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 2 & -1 & 1 \\ 0 & 7 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 2 & -1 & 1 \\ 0 & 1 & \frac{1}{7} \end{array}\right]$$

Last row $\Rightarrow x_2 = \frac{1}{7}$

1st row $\Rightarrow 2x_1 - \frac{1}{7} = 1 \Rightarrow x_1 = +\frac{8}{14}$

So $\vec{x} = \begin{bmatrix} +\frac{4}{7} \\ \frac{1}{7} \end{bmatrix}$

6. (a) (8 pts) Find all solutions to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Write your answer in vector form.

Row reduce: $\left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$

Interpret: x_3 is free: $x_3 = t$.

Last eqn $\Rightarrow -2x_4 = 0 \Rightarrow x_4 = 0$

2nd row $\Rightarrow 1 \cdot x_2 + 2x_3 + 0x_4 = 0$

ie $x_2 = -2x_3 = -2t$

1st row $\Rightarrow x_1 + 2x_2 + 0x_3 + 1 \cdot x_4 = 0 \Rightarrow x_1 = 4t + 0 = 4t$

$\therefore \vec{x} = t \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

(b) (2 pts) What is the dimension of the set of solutions to part (a)?

1

7. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

(a) (6 pts) Find A^{-1} using the Gauss-Jordan (row reduction) technique.

$$\rightarrow \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \xrightarrow{R2+2R3} \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \xrightarrow{R1-2R3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -1 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

(b) (2 pts) Check your answer by multiplying AA^{-1}.

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ -1 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

8. (3 pts) Write the vector equation $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ in the form $A\vec{x} = \vec{b}$.

That is, identify A, $\vec{x}$ and $\vec{b}$. Do not solve.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & -2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

9. (6 pts) Evaluate the following determinant. Show your work.

$$\begin{vmatrix} 1 & 3 & -1 & 2 \\ 1 & 0 & 3 & 0 \\ 2 & 1 & -1 & -1 \\ 0 & 2 & -2 & 0 \end{vmatrix}$$

Expand along 2^{nd} row:

$$= (-1)\,1 \begin{vmatrix} 3 & -1 & 2 \\ 1 & -1 & -1 \\ 2 & -2 & 0 \end{vmatrix} + 0 \left| \cdots \right| - 3 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix} + 0 \left| \cdots \right|$$

$$= \underbrace{(-1)\big((0+2-4) - (-4+6+0)\big)} + 0 + \underbrace{(-3)\left(0 \cdot |\cdot| - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 0|\cdots|\right)} + 0$$

$$= \cancel{0+4-6} \; \underline{+2} - 4 + 6 \qquad -30$$

$$= 4 \qquad - 30 = \underline{-26}$$

10. (6+1 pts) Give an example of a 2×2 matrix A and a vector $\vec{b}$ for which $A\vec{b} = \vec{0}$, but the entries of A are not all zero, and the entries of $\vec{b}$ are not all zero. Bonus point if no entry of A is zero and no entry of $\vec{b}$ is zero.

$$A\vec{b} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

11. Let $W = \left\{ \begin{bmatrix} n \\ 0 \end{bmatrix} \in \mathbb{R}^2 : n \in \mathbb{Z} \right\}$. Recall that $\mathbb{Z}$ is the set of all integers, or whole numbers: $\{.., -2, -1, 0, 1, 2, 3, ...\}$.

(a) (4pts) Is W closed under vector addition? Explain briefly.

Yes. $\begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} n+m \\ 0 \end{bmatrix}$. $\quad n, m \in \mathbb{Z} \Rightarrow n+m \in \mathbb{Z}$

(b) (4 pts) Is W closed under scalar multiblication? Explain briefly.

No. $\frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \notin W$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$.

(c) (2pts) Is W a vector subspace of $\mathbb{R}^2$? Justfy briefly.

No. W Not closed under ~~vector~~ scalar multiplication.

12. (6 pts) (True or False) $\begin{bmatrix}4\\6\\3\end{bmatrix}$ is in the span of $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Justify using the definition of span.

Solve $c_1\begin{bmatrix}2\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}4\\6\\3\end{bmatrix}$

i.e, $\begin{bmatrix}2&0\\1&1\\0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\6\\3\end{bmatrix}$

Row reduction:

$\begin{bmatrix}2&0&4\\1&1&6\\0&1&3\end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix}1&1&6\\2&0&4\\0&1&3\end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix}1&1&6\\0&-2&-8\\0&1&3\end{bmatrix}$

$\xrightarrow{R2+2R3} \begin{bmatrix}1&1&6\\0&0&-2\\0&1&3\end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix}1&1&6\\0&1&3\\0&0&-2\end{bmatrix}$ Last row $\Rightarrow$ $0c_1+0c_2=-2$

$\therefore$ No solution

13. (10 pts) Let $\mathcal{P}_1 = \{a+bx : a,b \in \mathbb{R}\}$. It turns out that $\mathcal{P}_1$ is a subspace of the set of all functions (with domain all real numbers and range in the real numbers). Show that the set $\{1, x+1\}$ is a basis for $\mathcal{P}_1$. Work directly from the definitions of linear independence and span.

1.) Lin. indep.: Solve $c_1 \cdot 1 + c_2 \cdot (x+1) = 0$

$\Rightarrow$ by differentiation $0c_1 + c_2 \cdot 1 = 0$

2^{nd} eq. $\Rightarrow$ $c_2 = 0$, 1^{st} eq becomes $1 \cdot c_1 + 0 \cdot (x+1) = 0$

i.e, $c_1 = 0$.

$\therefore$ $c_1 = 0$ and $c_2 = 0$ $\Rightarrow$ $\{1, x+1\}$ is a lin indep. set.

2.) Span: Solve $c_1 \cdot 1 + c_2 \cdot (x+1) = a+bx$ ($a+bx$ is an arbitrary function in P_1)

$\Rightarrow (c_1+c_2) + c_2 x = a+bx$

$\Rightarrow c_1 + c_2 = a$, $c_2 = b$

$\therefore c_1 + b = a \Rightarrow c_1 = a-b$

i.e, $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}a-b\\b\end{bmatrix}$.

$\therefore$ Any $a+bx$ is in the span of $\{1, x+1\}$.

i.e, $\text{Span}\{1, x+1\} = P_1$.

1) and 2) $\Rightarrow$ $\{1, x+1\}$ is a basis for P_1.

14. Extra credit (6 pts) Consider the vector equation $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix. Let W be the subset of all solutions to this vector equation. Assume that $\vec{y}$ is a function in W. Show that $c\vec{y}$ is also in W, where c is any real number.

i.e, vector

$\vec{y} \in W \Rightarrow A\vec{y} = \vec{0}$.

$\therefore A(c\vec{y}) = c(A\vec{y}) = c \cdot \vec{0} = \vec{0}$

$\therefore c\vec{y} \in W$. //

Math 3280, Differential Equations with Linear Algebra

Test 3

B. Peckham

November 29, 2018

Name ____

SHOW ALL WORK.
Please do not write on the provided Laplace Transform tables.
Indicate clearly any places where you use either the Laplace transform tables or a calculator.

p2.	______/28
p3.	______/34
p4.	______/25
p5.	______/13
EC	______/ 8
Total	______/100+8

1. (9 pts) Find the general solution to $y'' - 3y' - 4y = 2e^{5x}$. Do not use Laplace transforms.

For y_h: Try e^{rx}, Plug in: $e^{rx}(r^2 - 3r - 4) = 0$

3 $\Rightarrow r = -1, 4$

1 $\Rightarrow y_h(x) = c_1 e^{-x} + c_2 e^{4x}$

For y_p, try $Ae^{5x} \Rightarrow A \cdot 25e^{5x} - 3A(5e^{5x}) - A4(e^{5x}) = 2e^{5x}$

3 2 i.e., $A(25 - 15 - 4) = 2 \Rightarrow A = \frac{2}{6} = \frac{1}{3}$

So

$y = y_h + y_p$

$= c_1 e^{-x} + c_2 e^{4x} + \frac{1}{3} e^{5x}$

1

2. (6 pts) Find the general solution to the constant coefficient linear homogeneous differential equation which has the following characteristic polynomial: $(r^2 + 9)(r - 4)(r + 4)^2$ (obtained by trying a solution of the form $y(x) = e^{rx}$).

$y(x) = c_1 e^{4x} + c_2 e^{-4x} + c_3 x e^{-4x} + c_4 \cos 3x + c_5 \sin 3x$

3. For each function listed below, write down a constant coefficient differential operator that annihilates it. Use the notation $D = \frac{d}{dx}$.

(a) (2 pts) e^{3x}

$D - 3$

Ck: $(D-3)[e^{3x}] = 3e^{3x} - 3 \cdot e^{3x} = 0$ ✓

(b) (3 pts) $2e^{3x} + 4e^{-4x}$

$(D-3)(D+4)$

or, equivalently, $(D+4)(D-3)$

or $D^2 + D - 12$

4. Find the <u>form</u> of a particular solution to the following differential equations. Do not include any terms that are part of the complementary (homogeneous) solution, and do not evaluate the "undetermined coefficients."

(a) (4 pts) $y'' + y = \sin(t)$

$y_c = c_1 \cos t + c_2 \sin t$

$\Rightarrow y_p = At\cos t + Bt\sin t$

(b) (4 pts) $y'' + y = \sin(2t)$.

$y_p = A\sin 2t + B\cos 2t$

$y'' + y' + y = 4t^3 \Rightarrow y_p = At^3 + Bt^2 + Ct + D$

($r^2 + r + 1$ does not have a root of $r = 0$, so no "extra" multiples of t are needed)

$y'' - y' - 2y = 3\sin 2t + e^{2t} + 7e^{4t}$

From R.H.S. only, try $y_p = A\sin 2t + B\cos 2t + Ce^{2t} + De^{4t}$

For y_c: $r^2 - r - 2 = (r-2)(r+1) \Rightarrow e^{2t}$ is a sln for y_h.

$\therefore$ adjust to $y_p = A\sin 2t + B\cos 2t + \underline{Cte^{2t}} + De^{4t}$

5. What is the Laplace transform of the following functions. You may use the tables. You need not simplify your answer. ($u(t)$ is the unit step function.)

$g(t) =$

(a) (3 pts) $3t^4 - u(t-4) + e^{4t}\cos(2t)$?

$$G(s) = 3 \cdot \frac{4!}{s^5} - \frac{e^{-4s}}{s} + \frac{s-4}{(s-4)^2+2^2}$$

$g(t) =$

(b) (3 pts) $u(t-3)e^t$. Show your work.

$u(t-3)e^t = u(t-3)\, f(t-3)$ where $f(t-3) = e^t \Rightarrow f(t) = f((t+3)-3) = e^{t+3} = e^3 \cdot e^t$

$\therefore F(s) = e^3 \frac{1}{s-1}$. Table $\Rightarrow G(s) = e^{-3s} \cdot \frac{e^3}{s-1}$

6. (8 pts) Compute the Laplace transform of $f(t) = e^{2t}$ **directly from the definition of the Laplace transform** (not the tables).

$$\mathcal{L}\{e^{2t}\}(s) = \int_0^\infty e^{-st} e^{2t}\,dt = \int_0^\infty e^{-(s-2)t}\,dt = \left.\frac{e^{-(s-2)t}}{-(s-2)}\right|_0^\infty = 0 - \frac{1}{-(s-2)} = \frac{1}{s-2}$$

if $s > 2$

7. (6 pts) Define $f(t) = \begin{cases} t^2 & 0 \le t < 3 \\ 2-3t & 3 \le t < 5 \\ t^3 & 5 \le t. \end{cases}$ Use step functions to write $f(t)$ as a single line formula.

$$f(t) = t^2 + u(t-3)\left((2-3t) - t^2\right) + u(t-5)\left(t^3 - (2-3t)\right)$$

8. (8 pts) Solve using the method of Laplace transforms: $y'(t) = 4y(t),\ y(0) = 3$.

Transform: $sY(s) - y(0) = 4Y(s)$

ie, $sY(s) - 3 = 4Y(s)$.

Solve for $Y(s)$: $Y(s)(s-4) = 3$

$Y(s) = \frac{3}{s-4}$

untransform: $y(t) = 3e^{4t}$

9. (6 pts) Compute the Laplace transform of the solution of the initial value problem:

$$y'' + 3y' + 4y = 2e^{3x};\quad y(0) = 0, y'(0) = 3$$

(Find only $Y(s)$, not $y(t)$.) ~~Write your answer as a polynomial (in s) over a polynomial.~~ You need not simplify your answer.

Transform: $\left(s^2 Y(s) - s y(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) + 4Y(s) = 2\frac{1}{s-3}$

ie, $s^2 Y(s) - s \cdot 0 - 3 \quad + 3(sY(s) + 0) + 4Y(s) = \frac{2}{s-3}$

ie, $Y(s)(s^2 + 3s + 4) = 3 + \frac{2}{s-3}$

$$\Rightarrow \quad Y(s) = \frac{3}{s^2+3s+4} + \frac{2}{(s-3)(s^2+3s+4)} = \frac{3(s-3)+2}{(s-3)(s^2+3s+4)}$$

10. Find the inverse Laplace transform of

(a) (4pts) $G(s) = \frac{3s-1}{s^2+8s+20}$. $= \frac{3s-1}{(s+4)^2+2^2} = \frac{A(s+4)+B\cdot 2}{(s+4)^2+2^2}$, where $3s-1 = A(s+4)+2\cdot B$

ie, $3 = A$

$-1 = 4A+2B = 12+2B$

$\Rightarrow -13 = 2B$

or $B = \frac{-13}{2}$

$\therefore g(t) = 3e^{-4t}\cos 2t - \frac{13}{2}e^{-4t}\sin 2t$

(b) (4pts) $G(s) = e^{-2s}\frac{2}{s+4}$ $= e^{-2s}F(s)$ where $F(s) = \frac{2}{s+4}$

$\Rightarrow f(t) = 2e^{-4t} \Rightarrow f(t-2) = 2e^{-4(t-2)}$

Table $\Rightarrow$ $g(t) = u(t-2)f(t-2) = \underline{u(t-2)\,2e^{-4(t-2)}}$

11. (7 pts) Write the differential equation $u'' - 3u' + u = 1 - \cos(3t)$ with initial conditions $u(0) = 3$, $u'(0) = 4$ as an equivalent system of first order differential equations. Write the system in vector form: $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$. Write the initial conditions in vector form as well.

(4) Let $\underline{v = u'}$. Then $u'' - 3u' + u = 1-\cos(3t)$ becomes $\underline{v' - 3v + u = 1 - \cos 3t}$

Rewriting the underlined eqns: $u' = v$

$v' = -u + 3v + 1 - \cos 3t$

(3) Let $\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1-\cos 3t \end{pmatrix}$ then the system of 2 de's is equivalent to $\vec{x}' = A\vec{x} + \vec{b}$, with initial conditions: (1) $\vec{x}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

12. (10 pts) Find the solution to the vector differential equation $\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\vec{x}(t)$ with $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. You may use the fact that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are both eigenvectors for $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

(3) evals Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+-2 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 2$ is the eigenvalue for eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 + -2 \\ 2\cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \lambda = 3$ is the eigenvalue for eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(2) $\Rightarrow e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are both (independent) solutions to $\vec{x}' = A\vec{x}$.

(2) $\therefore$ Gen sln is $c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(3) I.C.'s $\Rightarrow \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. ie, $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

or $c_1 + 2c_2 = 1$

$c_1 + c_2 = 2$

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 1 \end{array}\right] \Rightarrow c_2 = -1$, $1c_1 + 2c_2 = 1$

$\Rightarrow c_1 = 1 - 2c_2 = 1+2 = 3$

$\therefore \vec{x}(t) = 3e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1\cdot e^{3t})\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

or $= \begin{bmatrix} 3e^{2t} - 2e^{3t} \\ 3e^{2t} - e^{3t} \end{bmatrix}$

13. (7 pts) Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. Find the eigenvalues for A and any one nonzero eigenvector for A.

Evals: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 9 = (\lambda - 1)^2 + 3^2 = 0$

if $(\lambda-1)^2 = -3^2$

$\Rightarrow (\lambda - 1) = \pm 3i$

or $\lambda = 1 \pm 3i$

For $\lambda = 1+3i$

Solve $(A - \lambda I)\vec{v} = \vec{0}$ w/ $\lambda = 1+3i$

i.e. $\begin{bmatrix} 1-(1+3i) & -3 \\ 3 & 1-(1+3i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. $\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -3i v_1 - 3v_2 = 0$ i.e. $v_2 = -iv_1$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ (or any multiple of this)

14. Consider the system of differential equations

$$\dot{x} = x - x^2 - xy, \quad \dot{y} = -\frac{1}{2}y$$

(a) (4 pts) Find all equilibrium points.

Eq pts: $\dot{x} = 0$ and $\dot{y} = 0$ i.e. $x - x^2 - xy = 0$, $-\frac{1}{2}y = 0 \Rightarrow y = 0$ $\Big\} \Rightarrow x - x^2 - x \cdot 0 = 0$

$\Rightarrow x - x^2 = 0$

or $x(1-x) = 0$

$\Rightarrow x = 0$ or 1

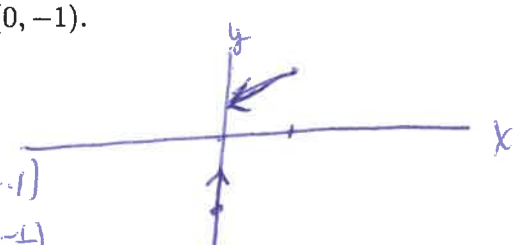
$\therefore (x,y) = (0,0)$ or $(1,0)$

(b) (2 pts) Compute and sketch in the (x, y) phase plane the two velocity vectors at $(x, y) = (1, 1)$ and $(x, y) = (0, -1)$.

@ $(x,y) = (1,1)$,

$(\dot{x}, \dot{y}) = (1 - 1^2 - 1 \cdot 1, -\frac{1}{2} \cdot 1)$

$= (-1, -\frac{1}{2})$



@ $(x,y) = (0,-1)$, $(\dot{x}, \dot{y}) = (0 - 0^2 - 0(-1), -\frac{1}{2}(-1)) = (0, \frac{1}{2})$

15. (Extra credit 8 pts) **Use the definition of the Laplace transform** to show that if $F(s)$ is the Laplace transform of $f(t)$, then the transform of $f'(t)$ is $sF(s) - f(0)$.

$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt = (0 - f(0)) + s\int_0^\infty e^{-st} f(t)\,dt$

let $u = e^{-st} \Rightarrow du = -se^{-st}$

$dv = f'(t)\,dt \quad v = f(t)$

$= -f(0) + sF(s)$

$= F(s) - f(0).$