Differential Equations with Linear Algebra, Math 3280
Lab #9: Systems of Differential Equations: Phase plane solutions

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Directions: Do the following tasks. Either provide the Lab Instructor with a writeup of your results, or have the Instructor check off each task you do. Your previous Spring-Mass Lab may be useful for problem 2. Choice of scale may be crucial, especially in problems 2 and 3.

Grading: 10 points per problem, 20 points total. 10 points extra credit. Details for grading points are indicated below.

1. A linear system of two equations. Consider the system

$$\dot{x} = -y, \dot{y} = x, \quad x(0) = 1, y(0) = 0.$$

(3pts) Analytic solution: an analytic solution can be obtained by the "eigenvalue-eigenvector" method. Do this to find the general solution to the differential equation, and the specific solution for the initial value problem (IVP).
(As a check, you should obtain the specific solution: $x(t) = \cos(t), y(t) = \sin(t)$. Show the work to obtain this answer.)

(3pts) Single numerical solution: determine a numerical solution to the same IVP using the *Mathematica* command NDSolve. (Check syntax of NDSolve for systems.) Make the following plots of the numerical solution: $x(t)$, $y(t)$, and y vs x. (The y vs x plot is the *phase plane.*) For the phase plane plot, you will need to use the command ParametricPlot.

(4pts) Phase plane: Use the StreamPlot command to plot multiple phase curves at once. On the output of the StreamPlot, locate the point corresponding to the initial conditions and highlight the corresponding phase curve. Sketch the velocity vector corresponding to the initial conditions. Locate and label any equilibria.

2. A second order linear differential equation converted to a system of differential equations. The differential equation we investigated for the spring-mass lab was

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = F_0\cos(\gamma t) \tag{1}$$

When converted to a system (make sure you know how to do this), it becomes

$$\dot{x} = y, \dot{y} = -2x - .4y + \frac{F_0}{25}\cos(\gamma t) \tag{2}$$

(a) (2pts) Choose parameter values of $F_0 = 0$ and $\gamma =$ anything. Numerical solutions: compute two numerical solutions using NDSolve and any two sets of initial conditions you would like to specify. Plot the two (projections of) solutions in the (x, y) phase plane.
(1pt) Are these solutions consistent with what you would expect for a spring-mass system? Explain.
(2pts) Sketch by hand, or use *Mathematica* to plot the vector field or direction field in the (x, y) plane. By hand, sketch several phase curves on the vector field plot/sketch. Compare with the numerically obtained phase curves.

(b) Now consider the full nonhomogeneous differential equation with $F_0 \neq 0$.
Analytic solution: given in spring-mass lab. Recall from Lab 7 the particular solution to equation (1) was

$$x_p(t) = \frac{F_0}{100 - 96\gamma^2 + 25\gamma^4}[(2 - \gamma^2)\cos(\gamma t) + \frac{2}{5}\gamma\sin(\gamma t)].$$

(1pt) Plot this solution, $x_p(t)$, in the (x, t) plane, its deriviative, $y(t) = x_p'(t)$ in the (y, t) plane, and the two together in the (x, y) plane, using ParametricPlot. Use $F_0 = 4$ and $\gamma = 1.386$.(the resonant frequency from Lab 7).
(2pts) Numerical solutions: For $F_0 = 4$ and $\gamma = 1.386$, and initial conditions $(x(0), y(0)) = (8, 0)$, use NDSolve to obtain a numerical solution to eq. (2) and then plot solution the three views: $x(t), y(t)$, and $(x(t), y(t))$. Compare the phase plane plot to the phase plane plot of the analytical solution. Note that this numerical solution is the full solution, not just the particular part of the solution.
(1pt) Why doesn't it matter (much) what initial conditions you choose? (Hint: Recall from the spring-mass lab what happens to the homogeneous part of the solution as $t \to \infty$?)
(1pt) Why am I not asking you to do a StreamPlot or look for equilibria?

3. Extra Credit: A nonlinear system. Assume a Rabbit and Fox population, measured in 100's, behaves according to the differential equations:

$$\dot{R} = R - R^2 - \frac{RF}{.25 + R}, \quad \dot{F} = -0.5F + \frac{FR}{.25 + R}$$

None of our analytic solution techniques work!! But we can still understand a lot about the behavior of solutions.
(2pts) Find all equilibria. (Hint: there are three.)
(2pts) Compute and display phase curves in the (R, F) plane using the 'StreamPlot' command in Mathematica.
(1pt) Determine the velocity vector at $(R, F) = (1, 1)$. Verify that your velocity vector is consistent with the StreamPlot.
(1pt) Use the Manipulate command to change the PlotRange for both the R and F variables and focus on the region in the phase space near the equilibria.
(4pts) Determine and describe the long-term bahavior of the Rabbit and Fox populations and how this behavior depends on the initial populations. You can restrict your answers to the first quadrant where the populations are nonnegative.