Math 3280

Differential Equations with Linear Algebra

Test 1

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Name A.K.

SHOW ALL WORK.

Calculators may be used for algebra and graphing only. You may NOT use calculators to solve differential equations (using commands like DSolve) or to take integrals or derivatives (using commands like D or Integrate).

p2.________/26

p3.________/30

p4.________/20

p5.________/24

EC.________/5

Total________/75

Directions: Do all problems. Show all work. Make no mistakes. Calculators may be used only for algebra and graphing, but not for symbolic tasks like integrating, differentiating, or solving differential equations. Implicit solutions will receive full credit only if labelled as **implicit**.

1. (4 pts) Consider the following differential equations. Are they separable, linear, both or neither? (Do not solve.)

   (a) $\dot{P} = 2P$ $\quad \dot{P} - 2P = 0$ lin. $\quad \frac{dP}{P} = 2\,dt$ sep

   (b) $\dot{P} = 2t$ $\quad \dot{P} - 0P = 2t$ lin $\quad dP = 2t\,dt$ sep

2. ~~(8 pts)~~ Solve the following two initial value problems. Explain your work.

   (a) (6 pts) $\dot{P} = 2P$, $P(0) = 2$ $\quad$ Inspection: $P(t) = P_0 e^{2t}$ $\quad P(0) = 2 \Rightarrow P(t) = 2e^{2t}$

   5 $\quad$ 1

   (b) (6 pts) $\dot{P} = 2t$, $P(0) = 2$ $\quad$ Integrate wrt t $\quad P(t) = \frac{2t^2}{2} + C = t^2 + C$ $\quad$ 5

   $P(0) = 3 \Rightarrow C = 2 \Rightarrow \underline{P(t) = t^2 + 2}$

   1

3. (10 pts) Find the general solution to $\dot{y} + 3y = 2e^{3t}$ and the specific solution when you add the initial condition $y(0) = 1$.

Linear (1st order) Let $\rho(t) = e^{\int 3\,dt} = e^{3t}$

$$\dot{y}e^{3t} + 3e^{3t}y = 2e^{3t}e^{3t} = 2e^{6t}$$

$$(ye^{3t})' = 2e^{6t}$$

Int w.r.t. t: $ye^{3t} = 2\frac{e^{6t}}{6} + C$

Explicit: $y(t) = \frac{e^{6t}}{3} \cdot e^{-3t} + Ce^{-3t}$

i.e. $y(t) = \frac{1}{3}e^{3t} + Ce^{-3t}$ (Gen sln.)

$y(0) = 1 \Rightarrow$

$\frac{1}{3} + C = 1 \Rightarrow C = \frac{2}{3}$

2 $\quad \therefore \underline{y(t) = \frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t}}$

(4 pts) Verify that your answer satisfies both the differential equation and the initial condition.

$y(t) = \frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t} \Rightarrow \dot{y}(t) = \frac{1}{3}3e^{3t} + \frac{2}{3}(-3)e^{-3t}$

3 $\quad = e^{3t} - 2e^{-3t}$

So $\dot{y} + 3y = (e^{3t} - 2e^{-3t}) + 3\left(\frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t}\right)$

$= 2e^{3t} + \left(-2 + \frac{2}{3}\cdot 3\right)e^{-3t} = 2e^{3t}$ ✓

1 $\quad$ IC: $y(0) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 1 = 1$ ✓

4. (10 pts) Find the general solution to $y' = (2x+1)/y$ and the specific solution when you add the initial condition $y(1) = 0$. $\boxed{y(2) = \sqrt{8}}$

$y \frac{dy}{dx} = (2x+1)\, dx$

$y(1) = 0 \Rightarrow \frac{0^2}{2} = 1^2 + 1 + C \Rightarrow C = -2$

$\therefore y(x) = \sqrt{2}\left(\pm\sqrt{x^2+x+C}\right)$ explicit general sln

$\Rightarrow \int y \frac{dy}{dx} dx = \int (2x+1)\, dx$

$\boxed{\frac{y^2}{2} = x^2 + x + C}$ (General Implicit)

$y(1)=0 \implies \frac{y^2}{2} = x^2 + x - 2$ (Specific Implicit) $\Rightarrow y = \pm\sqrt{2x^2+2x-4} = \sqrt{2}\left(\pm\sqrt{x^2+x-2}\right)$

(5 pts Extra Credit) What is the interval of existence for this initial value problem above?

Poorly worded problem. Ignore.

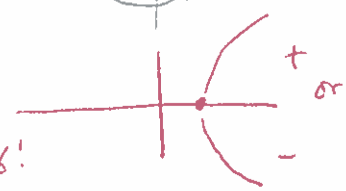
Need $x^2+x-2 \geq 0$, so

$= (x+2)(x-1)$

$\Rightarrow$ problems if $-2 < x < 1$

$\therefore x > 1$

But $y'(1)$ dne $\therefore$ int $= \emptyset$!



5. (10 pts) For what values of A and r is Ae^{rx} a solution to $y''(x) - 3y'(x) + 2y(x) = 0$?

$y = Ae^{rx} \Rightarrow y' = rAe^{rx}, \; y'' = r^2 Ae^{rx}$ 2

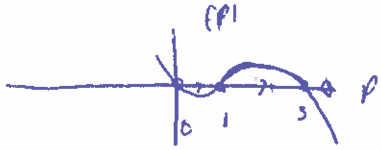
LHS : $Ae^{rx}(r^2 - 3r + 2) = 0 \Leftrightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$

2 2

$\Rightarrow r = 1, r = 2$ 2

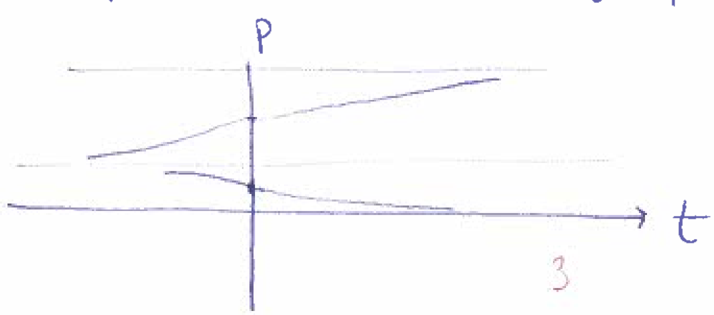
A anything 2

6. (10 pts) Sketch a phase line for the differential equation $\dot{P} = P(1-P)(P-3)$. Label your axis, indicate any equilibrium points with dots, and indicate the direction of growth/decline of the population (as time t increases) with arrows. Using only the information from the phase line, sketch possible solutions (both forward and backward in time) corresponding to the initial conditions $P(0) = 0.5$ and $P(0) = 2$. Label your axes. Describe the long-term (forward) fate of the population P for both initial conditions. (Do not solve the differential equation analytically.)



Phase line

5

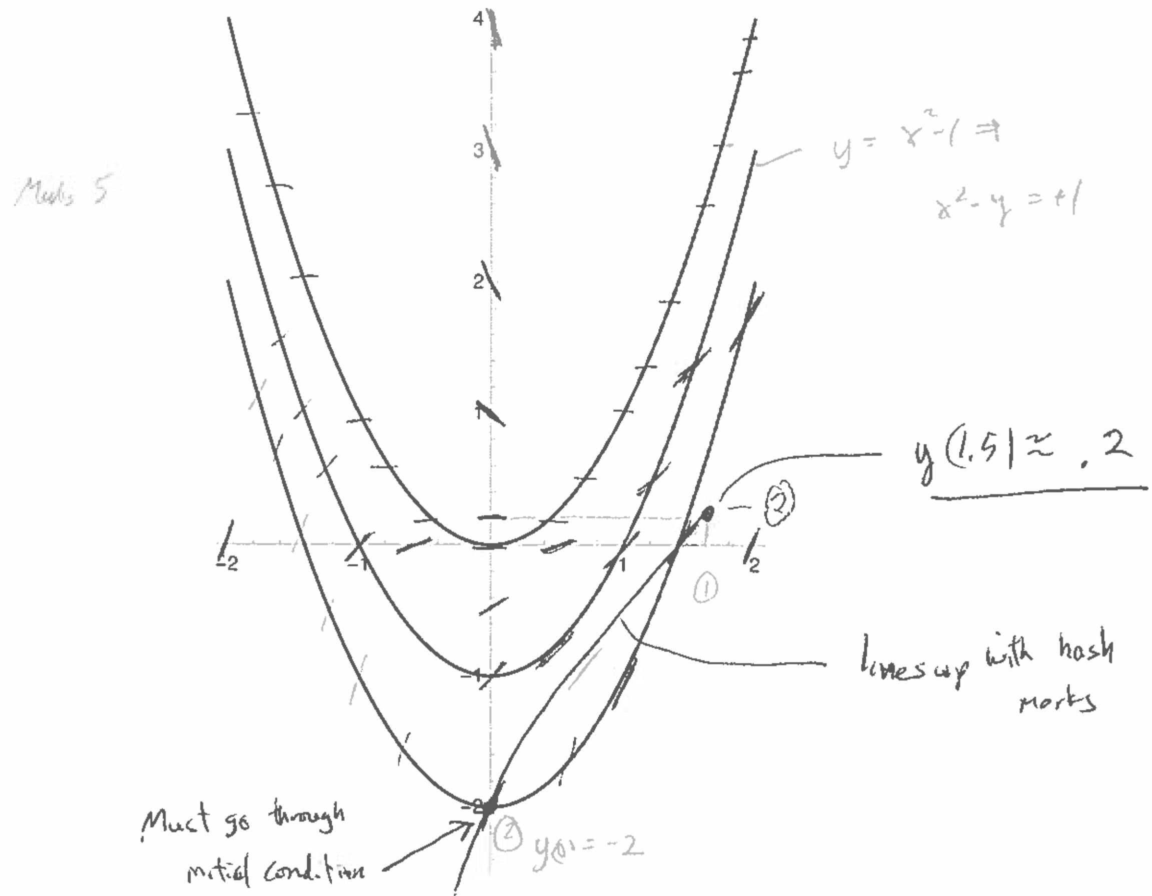


$P(0) = 2 \Rightarrow P(t) \to 3$ 1

$P(0) = .5 \Rightarrow P(t) \to 0$ 1

3

7. (10 pts) Sketch the portion of the slope field for $y'(x) = x^2 - y(x)$ along both axes and along the parabolas $y = x^2$, $y = x^2 - 1$ and $y = x^2 - 2$. (The three parabolas are shown for reference.) Include enough additional slope marks to allow you to sketch the solution corresponding to $y(0) = -2$. Use this sketch to estimate $y(1.5)$. (Do NOT solve analytically.)



8. (10 pts) Let $\phi(x)$ be the solution to the initial value problem $y'(x) = x^2 - y(x), y(0) = -2$. Numerically approximate $\phi(1.5)$ using Euler's method with a step size of 0.5. You do not need to simplify your answers. Do not solve analytically.

n	x_n	y_n
0	0	-2
1	.5	-1
2	1.0	$-.375$
3	1.5	$\boxed{.3125}$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$= -2 + (.5) \cdot (0 - (-2))$

$= -2 + 1 = -1$

$y_2 = -1 + (.5)((.5^2) - (-1))$

$= -1 + \frac{1.25}{2} = -1 + .625$

$= -.375$

$y_3 = -.375 + (.5)(1^2 + .375)$

$= -.375 + \frac{1.375}{2} = .3125$

9. Consider the differential equation $y' + y = \frac{1}{y^2}$. Make the substitution $v = y^3$ to replace y with v. Keep the independent variable as x. (8 pts) What is the differential equation in the new variables? (2 pts) Is this substitution useful in solving the original differential equation? Justify.

$v = y^3 \Rightarrow v' = 3y^2 y'$ 3

$3y^2 y' + y \cdot 3y^2 = 3y^2 \frac{1}{y^2}$ 3

8

Sub: $v' + 3v = 3$ 2

2 Yes: New eqn is linear, Orig. was non lin (But Sep !:)

10. Knowing that the rate of heating or cooling of hot coffee or cold icewater is proportional to the temperature difference between the drink and the (constant) room temperature T_0 leads to the following differential equation: $\dot{T} = k(T - T_0)$.

(a) (3pts) If the drink is coffee (hot), is $k > 0$ or $k < 0$? Explain briefly.

$\dot{T} < 0$, $T - T_0 > 0$ so $k < 0$

(b) (3pts) If the drink is icewater (cold), is $k > 0$ or $k < 0$? Explain briefly.

$\dot{T} > 0$, $T - T_0 < 0$, so $k < 0$

11. (10 pts) Assume that if unchecked, the population of mathematicians would grow at a rate proportional to the number of mathematicians alive at any given time. Assume, however, that there are two checks are operating. First, because of the interfering brain waves when too many mathematicians are too close together, mathematicians are eliminated at rate proportional to their population squared. (This is independent of the effect too many mathematicians have on the rest of humanity.) Secondly, in an effort to control the population of mathematicians the government has decided to eliminate them at a rate of 100 per year. If there are currently 10,000 living mathematicians, write a differential equation and initial condition, that describe the change in population as a function of time. Label any variables and constants you use; distinguish between them; indicate whether the constants should be positive or negative. (Do not solve.)

1 Let $M(t)$ = # mathematicians alive @ time t.

$\dot{M} = k_1 M - k_2 M^2 - 100$, $M(0) = 10000$

1 1 1 2

1 $k_1 > 0$ $k_2 > 0$

2 2 2