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Diff. Equations and Lin. Alg.
Math 3280
Quiz 10, Fall 2020
B. Peckham

1. (3 pts) Show directly from the definition of Laplace transform that if $F(s)$ is the Laplace transform of $f(t)$, then $-F'(s)$ is the Laplace transform of $tf(t)$.

$$-F'(s) = -\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = -\int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt = -\int_0^\infty -te^{-st}f(t)\,dt$$
$$= \int_0^\infty e^{-st}\left(t f(t)\right)dt = \mathcal{L}\{t f(t)\}(s)$$

2. Consider the function $F(s) = \dfrac{s}{s^2-6s+13}$.

(a) (2 pts) What is the partial fractional decomposition form for $F(s)$? Write your form so that the constants in your form are multiplied by functions of s that appear exactly in your Laplace transform tables.

$$F(s) = \frac{s}{(s-3)^2+4} = \frac{s}{(s-3)^2+2^2} = A\frac{(s-3)}{(s-3)^2+2^2} + B\frac{2}{(s-3)^2+2^2} = \frac{s-3}{(s-3)^2} + \frac{3}{2}\frac{2}{(s-3)^2+2^2}$$

$$\left[\text{where } s = A(s-3)+2B \Rightarrow A=1,\ 0=-3A+2B = -3+2B \Rightarrow B=\frac{3}{2}\right],\ \text{so}$$

(b) (2pts) Find the inverse Laplace transform of $e^{-2s}F(s)$. You need not solve for the constants in the partial fractional decomposition from part (a).

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = e^{3t}\cos 2t + \frac{3}{2}e^{3t}\sin 2t$$

$$\therefore \mathcal{L}^{-1}\{e^{-2s}F(s)\} \underset{\text{table}}{=} u(t-2)f(t-2) = u(t-2)\left(e^{3(t-2)}\cos 2(t-2) + \frac{3}{2}e^{3(t-2)}\sin 2(t-2)\right)$$

3. Consider the piecewise-defined function ~~in a single line, using the unit step function $u(t)$~~:

$$g(t) = \begin{cases} e^{2t} & 0 \le t < 4 \\ \sin(2(t-4)) & 4 \le t \end{cases}.$$

(a) (2pts) Write $g(t)$ in a single line, using the unit step function $u(t)$. $u(\,)$

$$g(t) = e^{2t} + u(t-4)\left(\sin 2(t-4) - e^{2t}\right)$$

(b) (1pt+2pts EC) Find the Laplace transform of $g(t)$.

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{s-2} + \mathcal{L}\{u(t-4)\sin 2(t-4)\}(s) - \mathcal{L}\{u(t-4)e^{2t}\}(s)$$

$$= \frac{1}{s-2} + e^{-4s}\frac{2}{s^2+2^2} - e^{-4s}\frac{e^8}{s-2}$$

Side work:

$$f_1(t-4) = \sin 2(t-4) \Rightarrow f_1(t) = \sin 2t \Rightarrow F_1(s) = \frac{2}{s^2+2^2}$$

$$f_2(t-4) = e^{2t} \Rightarrow f_2(t) = e^{2(t+4)} = e^8 e^{2t} \Rightarrow F_2(s) = \frac{e^8}{s-2}$$