Test 2 Solutions

1. Consider the differential equation $y'' + 10y' + 24y = 0$.

   (a) (8 pts) Find the general solution by guessing solutions of the form $y = e^{rx}$. Show your work from this guess. Try $y = e^{rx} \Rightarrow y' = re^{rx}, y'' = r^2e^{rx}$

   Plug in: $r^2e^{rx} + 10re^{rx} + 24e^{rx} = e^{rx}(r^2 + 10r + 24) = e^{rx}(r+6)(r+4) = 0$ if $r = -4, -6$

   $\therefore e^{-4x}$ and e^{-6x} are 2 (independent) solutions, so

   $$y(x) = c_1e^{-4x} + c_2e^{-6x}$$

   (b) (6 pts) Find one solution to the related nonhomogeneous differential equation: $y'' + 10y' + 24y = 5e^{2x}$ by guessing a soution of the form $y = Ae^{2x}$. Show your work.

   Try $y = Ae^{2x} \Rightarrow y' = 2Ae^{2x}, y'' = 4Ae^{2x}$

   Plug in: $4Ae^{2x} + 10(2Ae^{2x}) + 24Ae^{2x} = 5e^{2x}$

   Factor out e^{2x}: $4A + 20A + 24A = 5$

   ie, $48A = 3 \Rightarrow A = \frac{5}{48} = \cancel{\frac{1}{16}}$, so $y_p(x) = \frac{5}{48}e^{2x}$

   (c) (3 pts) Use (a) and (b) to determine the general solution to $y'' + 10y' + 24y = 5e^{2x}$? If you did not answer (a) or (b), indicate how you would use those answers to determine the answer to this problem.

   $$y(x) = y_c(x) + y_p(x) = c_1e^{-4x} + c_2e^{-6x} + \frac{1}{16}e^{2x}$$

2. (5 pts) Suppose one solution to $ay''(x) + by'(x) + cy(x) = 3\cos(x)$ is known to be $\sin(x)$. (a, b and c are constants which are not given.) Is $10\sin(x)$ also a solution to the same differential equation? Explain.

   No, if $y(x) = 10\sin x$, then $y'(x) = 10\cos x$, $y'' = -10\sin x$

   $\therefore ay'' + by' + cy = a(-10\sin x) + b\,10\cos x + c\,10\sin x$

   $= 10(a\,{-\sin x} + b\cos x + c\sin x) = 10(3\cos x) \neq 3\sin x$ (since $\sin x$ was a solution)

3. (6 pts) Assume the general solution to some second order differential equation is $y(x) = c_1e^{2x} + c_2e^{5x}$. What is the solution that also satisfies the initial conditions $y(0) = 0, y'(0) = 2$?

   $y(x) = c_1e^{2x} + c_2e^{5x} \Rightarrow y(0) = c_1 + c_2 = 0$

   $\Rightarrow y'(x) = 2c_1e^{2x} + 5c_2e^{5x} \Rightarrow y'(0) = 2c_1 + 5c_2 = 2$

   Solve for c_1, c_2

   1st equ $\Rightarrow c_1 = -c_2$. 2nd equ $\Rightarrow 2(-c_2) + 5c_2 = 2$, ie $3c_2 = 2$, $c_2 = \frac{2}{3}$, so $c_1 = -\frac{2}{3}$

   $\therefore y(x) = -\frac{2}{3}e^{2x} + \frac{2}{3}e^{5x}$

4. (5 pts) Let $A = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$. If they exist, compute AB and BA.

AB dne

$$BA = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 3 & 7 \\ 7 & 2 & 8 \end{bmatrix}$$

5. (6 pts) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Define a matrix E so that $EA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} - 2a_{21} & a_{32} - 2a_{22} & a_{33} - 2a_{23} \end{bmatrix}$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

6. (4 pts) If A is a 4×4 matrix, and $\det(A) = 3$, what is $\det(2A)$? Explain briefly.

$\det(2A) = 2^4 \cdot \det A = 2^4 \cdot 3 = \underline{48}$

Every entry multiplied by 2.
Every term in the determinant is the product of 4 ~~terms~~ entries.

7. (8 pts) Use row reduction to find all solutions to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Write your answer in vector form. Label row ops.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 2 & 5 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R3-2R1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow x_3 = t$ (free)
last eqn $\Rightarrow x_4 = 0$
2nd eqn $\Rightarrow x_2 = -2x_3 = \underline{-2t}$
1st eqn $\Rightarrow x_1 = -2x_2 - x_4 = -2(-2t) - 0 = \underline{4t}$

$$\therefore \vec{x} = t\begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

8. (6 pts) (True or False) $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Justify using the definition of span.

Solve $c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$.

Solve any method, eg, 3rd eqn $\Rightarrow c_2 = 3$
2nd eq. $\Rightarrow c_1 + c_2 = 2 \Rightarrow c_1 = 2 - c_2 = -1$
1st eq $\Rightarrow c_1 + c_2 = 4 \Rightarrow c_1 = 4 - c_2 = 1$

(or do row red.)

$\therefore$ no solution.
So $\underline{\text{False}}$

9. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

(a) (8 pts) Find A^{-1} using the Gauss-Jordan (row reduction) technique. Label all your row operations (like R2+3R1). show work.

$$[I : A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ -1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R1+2R2} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & -2 \\ -1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow[R2(-1)]{R3/3} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & -2 \\ 1 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R2-2R3]{R1+2R3} \left[\begin{array}{ccc|ccc} -1 & 2 & \frac{2}{3} & 1 & 0 & 0 \\ 1 & -1 & -\frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right]$$

$\underbrace{}_{A^{-1}}$

(b) (3 pts) Check your answer by multiplying AA^{-1}.

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 & \frac{2}{3} \\ 1 & -1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ✓$$

10. (4 pts) Write the vector equation $c_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\-5\end{bmatrix} + c_3 \begin{bmatrix}3\\4\\-2\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix}$ in the form $A\vec{x} = \vec{b}$.

That is, identify A, $\vec{x}$ and $\vec{b}$. Do not solve.

$$\begin{bmatrix}1 & 1 & 3\\2 & 1 & 4\\3 & 5 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}4\\2\\6\end{bmatrix}$$

11. (6 pts) Evaluate the following determinant. Show your work.

$$\begin{vmatrix}1 & 3 & -1 & 2\\5 & 0 & 0 & 0\\0 & 1 & -1 & -1\\1 & 2 & -2 & 0\end{vmatrix}$$

2^{nd} row expansion:

$$= -5\begin{vmatrix}3 & -1 & 2\\1 & -1 & -1\\2 & -2 & 0\end{vmatrix} + 6\begin{vmatrix}- & - & -\\- & - & -\\- & - & -\end{vmatrix} - 0\begin{vmatrix}1 & 3 & 2\\0 & 1 & -1\\1 & 2 & 0\end{vmatrix} + 0\begin{vmatrix}- & - & -\\- & - & -\\- & - & -\end{vmatrix}$$

1^{st} col. expansion

$$= -5\left(3\begin{vmatrix}-1 & -1\\-2 & 0\end{vmatrix} - 1\begin{vmatrix}-1 & 2\\-2 & 0\end{vmatrix} + 2\begin{vmatrix}-1 & 2\\-1 & -1\end{vmatrix}\right)$$

$$= (-5)(3(-2) - 1\cdot 4 + 2\cdot 3) = (-5)(-4) = \boxed{20}$$

$\cancel{-3\left(1\begin{vmatrix}1 & -1\\2 & 0\end{vmatrix} - 0 \quad + 1\begin{vmatrix}3 & 2\\1 & -1\end{vmatrix}\right)}$ $\cancel{= -5((-12) - 1\cdot 2 + 2\cdot 3)}$

$\cancel{= 3(1\cdot 2 + 1\cdot(-5))}$

$\cancel{= (-5)(-8) - 3(-3) = 40 + 9}$

$\cancel{= 49}$

12. (5pts) Give an example of a subset of $\Re^2$ that is closed under vector addition, but not under scalar multiplication. Explain briefly.

Lots of examples. One is the right half plane: $W = \left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0\right\}$.

- Adding any 2 vectors in W results in a vector with first coord. ≥ 0.
- $-1\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} \notin W$, even though $\begin{bmatrix}1\\0\end{bmatrix} \in W$.

13. Let $W = \{\vec{x} \in \Re^4 : A\vec{x} = \vec{0}\}$ for some constant matrix A. Suppose all solutions are $\vec{x} = t\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + r\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, where t, s, r can be any real numbers. Let $S = \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$.

(a) (3 pts) What must the dimensions of the matrix A be? Justify briefly.

A is $1 \times 4 \Rightarrow$ 1 equ, 4 unknowns – given from $\vec{x} \in \mathbb{R}^4$.
3 free $\Rightarrow$ only 1 leading var.

(b) (8 pts) Show directly from the definition that S is a linearly independent set.

Assume $c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Lots of ways to solve, but by inspection,
2^{nd} equ $\Rightarrow c_1 = 0$
3^{rd} equ $\Rightarrow c_2 = 0$
4^{th} equ $\Rightarrow c_3 = 0$

All c_i's $= 0 \Rightarrow S$ is lin independent

(c) (6 pts) Without doing any (further) computations, explain why S is a basis for W.

1. S is lin indep by part (b)
2. Span $S = W$ by the given sln: $\vec{x}$ is already written as the span of the 3 vectors in S.
1 and 2 $\Rightarrow$ S is a basis for W

(d) (EC +6pts) Figure out what the entries of the matrix A could be to have the given solution set W. Show your work.

The given sln $\Rightarrow x_2 = t,\ x_3 = s,\ x_4 = r,\ x_1 = t + s + r = x_2 + x_3 + x_4 \Rightarrow x_1 - x_2 - x_3 - x_4 = 0$

$\therefore A = \begin{bmatrix}1 & -1 & -1 & -1\end{bmatrix}$

9a

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1/3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$$

9b. Check: $A A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓