1. Consider the differential equation: $y'' - 3y' - 4y = 0$.

(a) (7pts) Solve by "guessing e^{rx}." Show all work. (Don't just jump to the polynomial.)

Let $y = e^{rx} \Rightarrow y' = re^{rx}$, $y'' = r^2 e^{rx}$

Plug in: $r^2 e^{rx} - 3(re^{rx}) - 4(e^{rx})$

$= e^{rx}(r^2 + (-3r) - 4)$

$= 0$ if $r^2 - 3r - 4 = (r-4)(r+1) = 0$

$\Rightarrow r = 4, -1$

$\therefore y(x) = C_1 e^{4x} + C_2 e^{-x}$

(b) (7 pts) Convert the differential equation above to a system. Write the system in vector form: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$. Define $\vec{x}(t)$. Do not solve.

Let $x_1 = y$

$x_2 = y' = x_1' \Rightarrow x_2' = y'' = 3y' + 4y = 4x_1 + 3x_2$

i.e., $x_1' = 0x_1 + 1x_2$
$x_2' = 4x_1 + 3x_2$

Vector form: $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \vec{x}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$

2. (7 pts) Find the general solution to the constant coefficient linear homogeneous differential equation which has the following characteristic polynomial: $((r+4)^2+9)(r+4)^2$ (obtained by trying a solution of the form $y(x) = e^{rx}$).

Polynomial is degree 4, so we need 4 basis functions for the Solution

$(r+4)^2 \Rightarrow$ double root of $-4 \Rightarrow e^{-4x}, xe^{-4x}$

$(r+4)^2+9 \Rightarrow$ complex roots $-4 \pm 3i \Rightarrow e^{-4x}\cos 3x, e^{-4x}\sin 3x$

$\therefore y(x) = C_1 e^{-4x} + C_2 x e^{-4x} + C_3 e^{-4x}\cos 3x + C_4 e^{-4x}\sin 3x$

3. For the following problems, assume the notation $D = \frac{d}{dx}$. each function listed below, write down a constant coefficient differential operator that annihilates it. Use the notation $D = \frac{d}{dx}$. Then operate on your function with your annihilator to show that you get the zero function.

(a) (5 pts) Show that $(D-3)^2$ annihilates $2xe^{3x}$ by operating on $2xe^{3x}$ with $(D-3)^2$.

$(D-3)^2[2xe^{3x}] = (D-3)(D-3)[2xe^{3x}] = (D-3)[2x3e^{3x} + 2e^{3x} - \cancel{3\cdot 2x}e^{3x}] = (D-3)[2e^{3x}]$

$= 3\cdot 2e^{3x} - 3\cdot 2e^{3x} = 0$

(b) (4 pts) What linear constant coefficient operator annihilates $\cos(2x)$? Make your operator as low order as possible.

$\cos 2x = e^{0x}\cos 2x \Rightarrow$ complex roots $0 \pm 2i \Rightarrow$ polynomial $(r-2i)(r+2i) = r^2+4$

$\therefore D^2 + 4$ annihilates $\cos 2x$.

4. Find the form of a particular solution to the following differential equations. Do not include any terms that are part of the complementary (homogeneous) solution, and do not evaluate the "undetermined coefficients." Explain your work briefly.

(a) (4 pts) $y'' - 7y' + 12y = x$. 1^{st} try: $y_p = Ax + B$. No term overlaps with Ax or B,

$\left[r^2 - 7r + 12 = (r-4)(r-3) \Rightarrow y_c = c_1 e^{4x} + c_2 e^{3x} \right]$ So stick with $\underline{y_p = Ax + B}$

(b) (5 pts) $y'' - 7y' + 12y = e^{2x} + e^{4x}$.

Again, $y_c = c_1 e^{4x} + c_2 e^{3x}$

1^{st} try for $y_p = Ae^{2x} + Be^{4x}$; Be^{4x} is a solution to the homogeneous de, so modify Be^{4x} to Bxe^{4x} $\therefore$ $y_p = \cancel{Ae^{2x} + Be^{4x}}$ $\underline{Ae^{2x} + Bxe^{4x}}$

5. What is the Laplace transform of the following functions. You may use the tables. You need not simplify your answer. ($u(t)$ is the unit step function.)

(a) (3 pts) $g(t) = te^{4t} - u(t-4) + e^{4t}\cos(2t)$?

$$G(s) = \frac{1}{(s-4)^2} - \frac{e^{-4s}}{s} + \frac{s-4}{(s-4)^2 + 2^2}$$

(b) (5 pts) $g(t) = u(t-3)e^t$. Show your work. Use $\mathcal{L}\, u(t-a) f(t-a) = e^{-as} F(s)$

where $a = 3$, $f(t-3) = e^t \Rightarrow f(t) = e^{t+3} = e^3 e^t \Rightarrow F(s) = e^3 \cdot \frac{1}{s-1}$

$$\therefore \mathcal{L}\{u(t-3)e^t\}(s) = \underline{e^{-3s} \frac{e^3}{s-1}}$$

6. (6 pts) Define $f(t) = \begin{cases} t^2 & 0 \le t < 3 \\ 0 & 3 \le t < 5 \\ t^3 & 5 \le t. \end{cases}$ Use step functions to write $f(t)$ as a single line formula.

$$f(t) = t^2 + u(t-3)(0 - t^2) + u(t-5)(t^3 - 0)$$

7. (8 pts) Solve using the method of Laplace transforms: $y'(t) = 4y(t) + 1$, $y(0) = 3$.

$sY(s) - 3 = 4Y(s) + \frac{1}{s}$

$Y(s)(s-4) = 3 + \frac{1}{s}$

$Y(s) = \frac{3 + \frac{1}{s}}{s-4} = \frac{3s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

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where $3s + 1 = A(s-4) + Bs$

$\Rightarrow 3 = A + B \Rightarrow A = -\frac{1}{4}$

$1 = -4A$ $\quad 3 = -\frac{1}{4} + B \Rightarrow B = 3\frac{1}{4} = \frac{13}{4}$

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$\therefore Y(s) = -\frac{1}{4}\frac{1}{s} + \frac{13}{4}\frac{1}{s-4}$

$\Rightarrow \underline{y(t) = -\frac{1}{4} \cdot 1 + \frac{13}{4}e^{4t}}$

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8. (6 pts) Compute the Laplace transform of the solution of the initial value problem. (Find only $Y(s)$, not $y(t)$.) You need not simplify your answer.

$$y'''(t) + y'' + 3y' + 4y = 2e^{3t}; \quad y(0) = 0, y'(0) = 3, y''(0) = 2$$

$$\left(s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)\right) + \left(s^2 Y(s) - s y(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) + 4Y(s) = \frac{2}{s-3}$$

$$\Rightarrow Y(s)\left(s^3 + s^2 + 3s + 4\right) - s \cdot 3 - 2 - 3 + 3(-0) = \frac{2}{s-3}$$

$$\Rightarrow Y(s)\left(s^3 + s^2 + 3s + 4\right) = 3s + 5 + \frac{2}{s-3} \Rightarrow Y(s) = \frac{3s+5+2/(s-3)}{s^3+s^2+3s+4}$$

or $= \frac{2 + (3s+5)(s-3)}{(s-3)(s^3+s^2+3s+4)}$

or $= \frac{3s^2-4s-13}{(s-3)(s^3+s^2+3s+4)}$

9. Find the inverse Laplace transform of

(a) (6 pts) $G(s) = \dfrac{3s+1}{s^2+8s+20}$. $= \dfrac{3s+1}{(s+4)^2+2^2} = A \dfrac{s+4}{(s+4)^2+2^2} + B \cdot \dfrac{2}{(s+4)^2+2^2}$

where $A(s+4) + B \cdot 2 = 3s+1$

$\Rightarrow \begin{cases} A = 3 \\ 4A + 2B = 1 \end{cases} \Rightarrow A = 3$

$4 \cdot 3 + 2B = 1 \Rightarrow B = -\frac{11}{2}$. So $g(t) = 3e^{-4t}\cos 2t - \frac{11}{2} e^{-4t} \sin 2t$

(b) (4 pts) $G(s) = e^{-2s} \dfrac{2}{s+5}$ $= e^{-2s} \dfrac{2}{s+5} = e^{-2s} F(s)$ where $F(s) = \dfrac{2}{s+5} \Rightarrow f(t) = 2e^{-5t}$

$\therefore$ using $\mathcal{L}\{u(t-a) f(t-a)\}(s) = e^{-as} F(s)$, we get

$$g(t) = u(t-2) f(t-2) = u(t-2) \cdot 2e^{-5(t-2)}$$

10. (6 pts) Consider the system of differential equations and initial conditions given by $\dfrac{d\vec{x}(t)}{dt} = A\vec{x}(t)$, with $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Assume you do not know the matrix A, but that you know it is 2×2, and has eigenvalues 3 and 2, with respective eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the solution to the initial value problem.

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

i.e., $\begin{aligned} c_1 + 2c_2 &= 1 \\ c_1 + c_2 &= 2 \end{aligned} \Rightarrow c_2 = 1 - 2 = -1$, $c_1 = 3$

$$\therefore \vec{x}(t) = 3e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

11. (7 pts) Let $A = \begin{bmatrix} +3 & -2 \\ 2 & +3 \end{bmatrix}$. Find the eigenvalues for A and any one (nonzero) eigenvector for A. Show all work.

$$\det(A - \lambda I) = \begin{vmatrix} +3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (+3-\lambda)(3-\lambda) + 2\cdot 2$$

$$= (\lambda - 3)^2 + 4 \Rightarrow \lambda - 3 = \pm 2i \quad \text{(or use quadratic formula)}$$

$$\text{or } \lambda = +3 \pm 2i$$

For $\lambda = 3+2i$, Solve $(A - \lambda I)\vec{v} = \vec{0}$

i.e. $\begin{pmatrix} +3-(3+2i) & -2 \\ 2 & +3-(3+2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

top equ $\Rightarrow -2i\, v_1 - 2v_2 = 0$

or $v_2 = -i\, v_1$

$\therefore \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ or any complex multiple, like $\begin{bmatrix} i \\ 1 \end{bmatrix}$

12. (7 pts) **Use the definition of the Laplace transform** to show that if $F(s)$ is the Laplace transform of $f(t)$, then the transform of $f'(t)$ is $sF(s) - f(0)$.

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\, dt$$

Let $u = e^{-st} \Rightarrow du = -se^{-st}\, dt$

$dv = f'(t)\, dt \quad v = f(t)$

$$= \left(\lim_{A\to\infty} e^{-sA} f(A) - e^{-s\cdot 0} f(0)\right) - (-s)\int_0^\infty e^{-st} f(t)\, dt$$

$$= 0 - f(0) + s F(s) = sF(s) - f(0)$$

13. Extra Credit (+6 pts). Define the differential operator L by $L = D^2 + 5D - 3$, where D is the derivative operator $\frac{d}{dx}$. Assume that $y_1(x)$ and $y_2(x)$ are known functions that are solutions to $L[y] = 0$. Show that any function of the form $c_1y_1(x) + c_2y_2(x)$ (where c_1 and c_2 are constants) is also a solution to $L[y] = 0$. You may use operator notation, but are not required to do so.

Using operator notation:

y_1, y_2 slns $\Rightarrow L[y_1] = 0,\ L[y_2] = 0$

$\therefore L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$

$= c_1\cdot 0 + c_2\cdot 0$

$= 0 + 0 = 0.$

$\therefore c_1y_1 + c_2y_2$ is a solution to $L[y] = 0.$

Using $L[y] = (D^2+5D-3)[y] = y'' + 5y' - 3y$

$\therefore L[y] = 0$ is $y'' + 5y' - 3y = 0$

y_1, y_2 Solutions $\Rightarrow y_1'' + 5y_1' - 3y_1' = 0 \quad (1)$

and $y_2'' + 5y_2' - 3y_2 = 0$

$\therefore (y_1+y_2)'' + 5(y_1+y_2)' - 3(y_1+y_2)$

$= (y_1'' + 5y_1' - 3y_1) + (y_2'' + 5y_2' - 3y_2')$

$= 0 + 0 = 0$ ✓ i.e. $y_1 + y_2$ is a solution to (1)